Today

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Finish Euclid.

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Bijection/CRT/Isomorphism.

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Fermat's Little Theorem.

Greatest Common Divisor and Inverses.

Thm:
If greatest common divisor of x and m, $\gcd(x, m)$, is 1, then x has a multiplicative inverse modulo m.

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Proof Sketch: The set $S = \{0x, 1x, \ldots, (m-1)x\}$ contains $y \equiv 1 \mod m$ if all distinct modulo m.

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For $x = 4$ and $m = 6$. All products of 4...
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Very different for elements with inverses.

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Poll

Which is bijection?
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(B) is not.

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Excursion: Value and Size.

Before discussing running time of gcd procedure...

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one million or 1,000,000!

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Poll.

Assume $\log_2 1,000,000$ **is 20 to the nearest integer.**
Mark what's true.

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Mark what's true.

(A) The size of 1,000,000 is 20 bits.
(B) The size of 1,000,000 is one million.
(C) The value of 1,000,000 is one million.
(D) The value of 1,000,000 is 20.

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(A) The size of 1,000,000 is 20 bits.
(B) The size of 1,000,000 is one million.
(C) The value of 1,000,000 is one million.
(D) The value of 1,000,000 is 20.

(A) and (C).

Poll

Which are correct?

(A) gcd(700,568) = gcd (568,132)
(B) gcd(8,3) = gcd(3,2)
(C) gcd(8,3) = 1
(D) gcd(4,0) = 4

Poll

Which are correct?

(A) gcd(700,568) = gcd (568,132)
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Algorithms at work.

Trying everything

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Trying everything

Check 2, check 3, check 4, check 5 $\ldots$, check $y/2$.

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Check 2, check 3, check 4, check 5 . . . , check $y/2$.

"(gcd x y)" at work.

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"(gcd x y)" at work.

```
euclid(700,568)
```

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```
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  euclid(568, 132)
```

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Check 2, check 3, check 4, check 5 . . . , check $y/2$.

"(gcd x y)" at work.

```
euclid(700,568)
  euclid(568, 132)
    euclid(132, 40)
```

Algorithms at work.

Trying everything

Check 2, check 3, check 4, check 5 . . . , check $y/2$.

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```
euclid(700,568)
  euclid(568, 132)
    euclid(132, 40)
      euclid(40, 12)
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Notice: The first argument decreases rapidly.

Algorithms at work.

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Notice: The first argument decreases rapidly.
 At least a factor of 2 in two recursive calls.

Algorithms at work.

Trying everything

Check 2, check 3, check 4, check 5 . . . , check $y/2$.

"(gcd x y)" at work.

```
euclid(700,568)
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    euclid(132, 40)
      euclid(40, 12)
        euclid(12, 4)
          euclid(4, 0)
            4
```

Notice: The first argument decreases rapidly.
At least a factor of 2 in two recursive calls.

(The second is less than the first.)

Runtime Proof.

```
(define (euclid x y)
 (if (= y 0)
     x
    (euclid  y (mod x y))))
```

Theorem: (euclid x y) uses $O(n)$ "divisions" where $n = b(x)$.

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After $2\log_2 x = O(n)$ recursive calls, argument x is 1 bit number.

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One more recursive call to finish.
1 division per recursive call.
$O(n)$ divisions.

$\square$

Runtime Proof (continued.)

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First arg decreases by at least factor of two in two recursive calls.

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Proof of Fact: Recall that first argument decreases every call.

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Fact:
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Case 1: $y < x/2$, first argument is y
$\implies$ true in one recursive call;

Runtime Proof (continued.)

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mod (x, y) is second argument in next recursive call,

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When $y \geq x/2$, then

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When $y \geq x/2$, then

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Runtime Proof (continued.)

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$\text{mod}\,(x,y)$ is second argument in next recursive call,
and becomes the first argument in the next one.

When $y \geq x/2$, then

$$\lfloor \frac{x}{y} \rfloor = 1,$$

$$\text{mod}\,(x,y) = x - y\lfloor \frac{x}{y} \rfloor =$$

Runtime Proof (continued.)

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Fact:
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$\Longrightarrow$ true in one recursive call;

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Runtime Proof (continued.)

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$$\text{mod}\,(x, y) = x - y\lfloor \frac{x}{y} \rfloor = x - y \leq x - x/2 = x/2$$

$\square$

Poll

Mark correct answers.
Note: Mod(x,y) is the remainder of x divided by y.

Poll

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Note: Mod(x,y) is the remainder of x divided by y.

(A) $\mod(x,y) < y$
(B) If euclid(x,y) calls euclid(u,v) calls euclid (a,b) then $a <= x/2$.
(C) euclid(x,y) calls euclid (u,v) means u = y.
(D) if $y > x/2, mod(x,y) < y/2$
(E) if $y > x/2, mod(x,y) = (y - x)$

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(D) if $y > x/2, mod(x,y) < y/2$
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(D) is not always true.

Finding an inverse?

We showed how to efficiently tell if there is an inverse.

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Extend euclid to find inverse.

Euclid's GCD algorithm.

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(define (euclid x y)
 (if (= y 0)
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Computes the gcd(x, y) in $O(n)$ divisions. (Remember $n = \log_2 x$.)

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For x and m, if $\gcd(x, m) = 1$ then x has an inverse modulo m.

Multiplicative Inverse.

GCD algorithm used to tell **if** there is a multiplicative inverse.

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GCD algorithm used to tell **if** there is a multiplicative inverse.

How do we **find** a multiplicative inverse?

Extended GCD

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$$ax + by$$

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$$ax \equiv 1 - bm \equiv 1 \pmod{m}.$$

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Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

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$$(3)12 + (-1)35 = 1.$$

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$a = 3$ and $b = -1$.

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$$\begin{aligned} ax + bm &= 1 \\ ax \equiv 1 - bm &\equiv 1 \pmod{m}. \end{aligned}$$

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Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$$(3)12 + (-1)35 = 1.$$

$a = 3$ and $b = -1$.
The multiplicative inverse of $12 \pmod{35}$ is 3.

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Check: 3(12)

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The multiplicative inverse of $12 \pmod{35}$ is 3.

Check: $3(12) = 36$

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The multiplicative inverse of $12 \pmod{35}$ is 3.

Check: $3(12) = 36 = 1 \pmod{35}$.

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
```

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
```

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
```

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

Make d out of multiples of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?

Make d out of multiples of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Make d out of multiples of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

Make *d* out of multiples of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

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But we want 1 from sum of multiples of 35 and 12?

Make d out of multiples of x and y..?

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  gcd(12, 11)  ;;  gcd(12, 35%12)
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But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.

Make *d* out of multiples of *x* and *y*..?

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Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35$

Make d out of multiples of x and y..?

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Get 11 from 35 and 12 and plugin....

Make d out of multiples of x and y..?

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gcd(35,12)
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But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35$
Get 11 from 35 and 12 and plugin.... Simplify. $a = 3$ and $b = -1$.

Extended GCD Algorithm.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
     else
         (d, a, b) := ext-gcd(y, mod(x,y))
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Claim: Returns (d,a,b): $d = gcd(a,b)$ and $d = ax + by$.

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Example:

```
ext-gcd(35,12)
  ext-gcd(12, 11)
```

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Example: $a - \lfloor x/y \rfloor \cdot b =$

```
ext-gcd(35,12)
  ext-gcd(12, 11)
    ext-gcd(11, 1)
      ext-gcd(1,0)
      return (1,1,0) ;; 1 = (1)1 + (0) 0
```

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```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
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```

Claim: Returns (d,a,b): $d = gcd(a,b)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b = 1 - \lfloor 11/1 \rfloor \cdot 0 = 1$

```
    ext-gcd(35,12)
      ext-gcd(12, 11)
        ext-gcd(11, 1)
          ext-gcd(1,0)
          return (1,1,0) ;; 1 = (1)1 + (0) 0
        return (1,0,1)   ;; 1 = (0)11 + (1)1
```

Extended GCD Algorithm.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
     else
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         return (d, b, a - floor(x/y)  * b)
```

Claim: Returns (d,a,b): $d = gcd(a,b)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b = \quad 0 - \lfloor 12/11 \rfloor \cdot 1 = -1$

```
ext-gcd(35,12)
  ext-gcd(12, 11)
    ext-gcd(11, 1)
      ext-gcd(1,0)
      return (1,1,0) ;; 1 = (1)1 + (0) 0
    return (1,0,1)   ;; 1 = (0)11 + (1)1
  return (1,1,-1)    ;; 1 = (1)12 + (-1)11
```

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```
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```

Claim: Returns (d,a,b): $d = gcd(a,b)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b =$ $1 - \lfloor 35/12 \rfloor \cdot (-1) = 3$

```
ext-gcd(35,12)
  ext-gcd(12, 11)
    ext-gcd(11, 1)
      ext-gcd(1,0)
      return (1,1,0) ;; 1 = (1)1 + (0) 0
    return (1,0,1)   ;; 1 = (0)11 + (1)1
  return (1,1,-1)    ;; 1 = (1)12 + (-1)11
return (1,-1, 3)     ;; 1 = (-1)35 +(3)12
```

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    ext-gcd(35,12)
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        return (1,0,1)   ;; 1 = (0)11 + (1)1
      return (1,1,-1)    ;; 1 = (1)12 + (-1)11
   return (1,-1, 3)      ;; 1 = (-1)35 +(3)12
```

Extended GCD Algorithm.

```
ext-gcd(x,y)
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         return (d, b, a - floor(x/y)  * b)
```

Theorem: Returns (d, a, b), where $d = gcd(a, b)$ and

$$d = ax + by.$$

Correctness.

Proof: Strong Induction.[1]

[1] Assume d is $gcd(x, y)$ by previous proof.

Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd$(x, 0)$ returns $(d = x, 1, 0)$ with $x = (1)x + (0)y$.

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Ind hyp: **ext-gcd**$(y, \mod(x,y))$ returns (d, a, b) with
$d = ay + b(\mod(x,y))$

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$$\begin{aligned} d &= ay + b \cdot (\mod (x, y)) \\ &= ay + b \cdot (x - \lfloor \frac{x}{y} \rfloor y) \end{aligned}$$

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$$\begin{aligned} d &= ay + b\cdot(\mod(x,y)) \\ &= ay + b\cdot(x - \lfloor \frac{x}{y} \rfloor y) \\ &= bx + (a - \lfloor \frac{x}{y} \rfloor \cdot b)y \end{aligned}$$

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And ext-gcd returns $(d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))$ so theorem holds!

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And ext-gcd returns $(d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))$ so theorem holds! $\square$

[1] Assume d is $gcd(x, y)$ by previous proof.

Review Proof: step.

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```

Recursively: $d = ay + b(x - \lfloor \frac{x}{y} \rfloor \cdot y)$

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Recursively: $d = ay + b(x - \lfloor \frac{x}{y} \rfloor \cdot y) \implies d = bx - (a - \lfloor \frac{x}{y} \rfloor b)y$

Returns $(d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))$.

Hand Calculation Method for Inverses.

Example: $\gcd(7, 60) = 1$.

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$$7(0) + 60(1) = 60$$

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Example: $\gcd(7,60) = 1$.
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$$
\begin{aligned}
7(0) + 60(1) &= 60 \\
7(1) + 60(0) &= 7
\end{aligned}
$$

Hand Calculation Method for Inverses.

Example: $\gcd(7,60) = 1$.
egcd(7,60).

$$\begin{aligned} 7(0) + 60(1) &= 60 \\ 7(1) + 60(0) &= 7 \\ 7(-8) + 60(1) &= 4 \end{aligned}$$

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Example: $\gcd(7,60) = 1$.
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$$\begin{aligned} 7(0)+60(1) &= 60 \\ 7(1)+60(0) &= 7 \\ 7(-8)+60(1) &= 4 \\ 7(9)+60(-1) &= 3 \end{aligned}$$

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$$\begin{aligned} 7(0)+60(1) &= 60 \\ 7(1)+60(0) &= 7 \\ 7(-8)+60(1) &= 4 \\ 7(9)+60(-1) &= 3 \\ 7(-17)+60(2) &= 1 \end{aligned}$$

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$$

Confirm:

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Confirm: $-119+120 = 1$

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Confirm: $-119+120=1$

Note: an "iterative" version of the e-gcd algorithm.

Wrap-up

Conclusion: Can find multiplicative inverses in $O(n)$ time!

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Very different from elementary school: try 1, try 2, try 3...

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$2^{n/2}$

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Inverse of $500,000,357$ modulo $1,000,000,000,000$?

Wrap-up

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Inverse of $500,000,357$ modulo $1,000,000,000,000$?
≤ 80 divisions.

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Inverse of $500,000,357$ modulo $1,000,000,000,000$?
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Internet Security.

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Public Key Cryptography: 512 digits.

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512 divisions vs.

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Public Key Cryptography: 512 digits.
512 divisions vs.
$(10000000000000000000000000000000000000000000)^5$ divisions.

Wrap-up

Conclusion: Can find multiplicative inverses in $O(n)$ time!

Very different from elementary school: try 1, try 2, try 3...

$2^{n/2}$

Inverse of $500,000,357$ modulo $1,000,000,000,000$?
≤ 80 divisions.
versus $1,000,000$

Internet Security.
Public Key Cryptography: 512 digits.
512 divisions vs.
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$x = 5 \pmod 7$ and $x = 3 \pmod 5$.

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A bit slow for large values.

Simple Chinese Remainder Theorem.

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Find $x = a \pmod{m}$ and $x = b \pmod{n}$

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Find $x = a \pmod{m}$ and $x = b \pmod{n}$ where gcd(m, n)=1.

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Let $x = au + bv$.

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$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
$x = b \pmod{n}$ since $au = 0 \pmod{n}$ and $bv = b \pmod{n}$
This shows there is a solution. $\square$

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If not, two solutions, x and y.

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Proof (uniqueness):
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$(x - y) \equiv 0 \pmod{m}$ and $(x - y) \equiv 0 \pmod{n}$.
$\implies (x - y)$ is multiple of m and n

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$\implies mn|(x-y)$

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&\qquad\qquad \implies mn|(x-y)
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$\implies x - y \geq mn \implies x, y \notin \{0, \ldots, mn-1\}$.

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$$\implies mn|(x-y)$$

$\implies x - y \geq mn \implies x, y \notin \{0, \ldots, mn-1\}$.
Thus, only one solution modulo mn.

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Poll.

My love is won,
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Nothing and nothing done.

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What is the rhyme saying?

(A) Multiplying by 1, gives back number. (Does nothing.)
(B) Adding 0 gives back number. (Does nothing.)
(C) Rao has gone mad.
(D) Multiplying by 0, gives 0.
(E) Adding one does, not too much.

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All are (maybe) correct.

Poll.

My love is won,
Zero and one.
Nothing and nothing done.

What is the rhyme saying?

(A) Multiplying by 1, gives back number. (Does nothing.)
(B) Adding 0 gives back number. (Does nothing.)
(C) Rao has gone mad.
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Mapping is "isomorphic":
corresponding addition (and multiplication) operations consistent with mapping.

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For a prime modulus, we can reduce exponents modulo $p-1$!

Lecture in a minute.

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Product of elts == for range/domain: a^{p-1} factor in range.