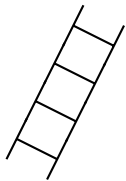


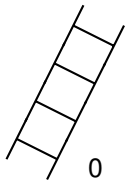
The natural numbers.

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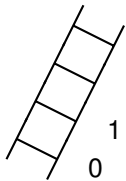
The natural numbers.

0,



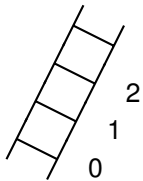
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0, 1,



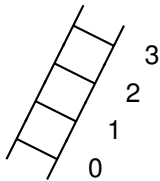
The natural numbers.

0, 1, 2,

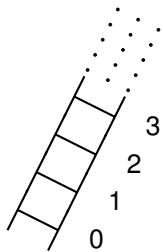


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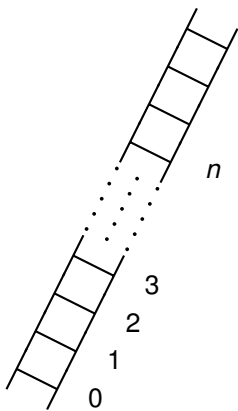


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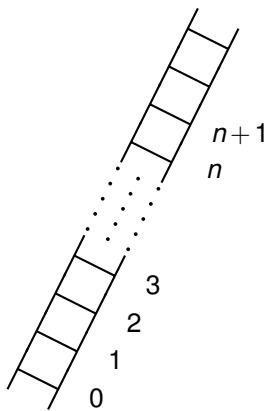
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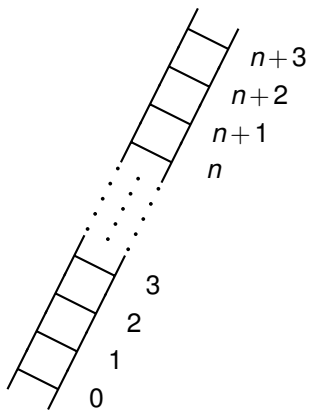
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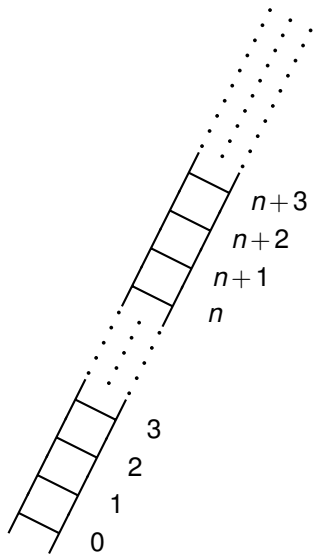
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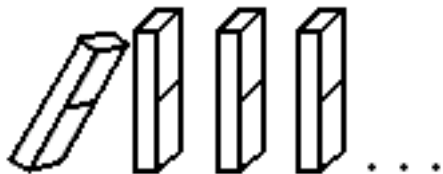
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Notes visualization

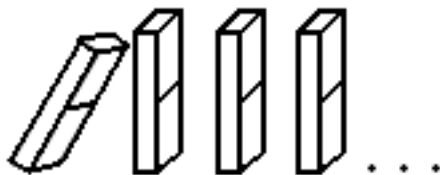
Note's visualization: an infinite sequence of dominos.



Prove they all fall down;

Notes visualization

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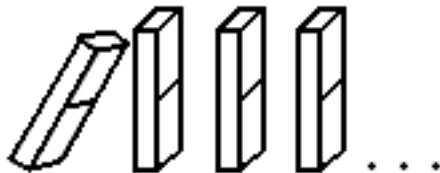


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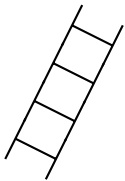


Prove they all fall down;

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“ k th domino falls implies that $k+1$ st domino falls”

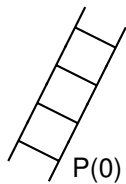
Climb an infinite ladder?

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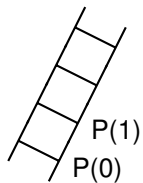


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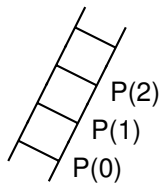


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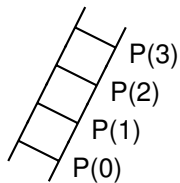
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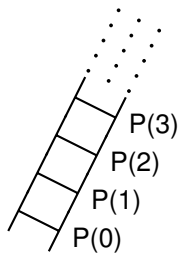
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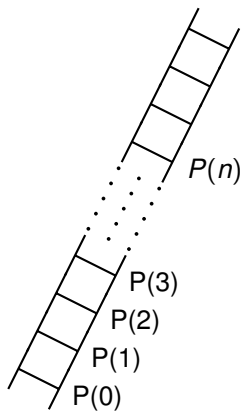
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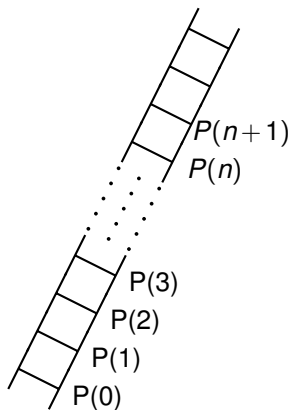
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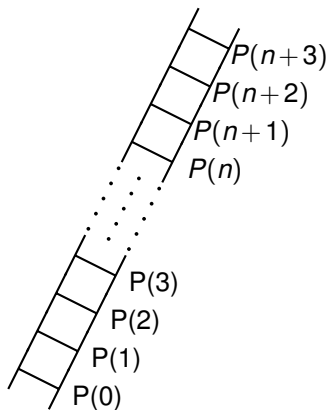
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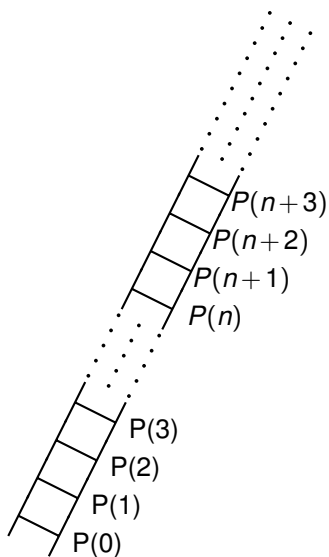
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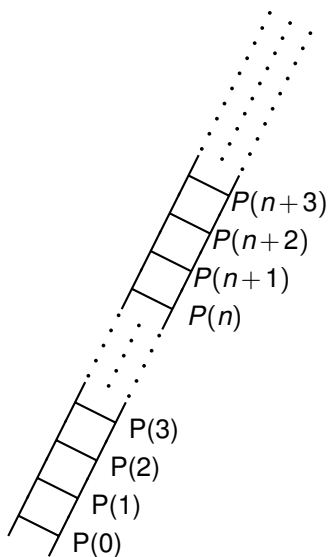
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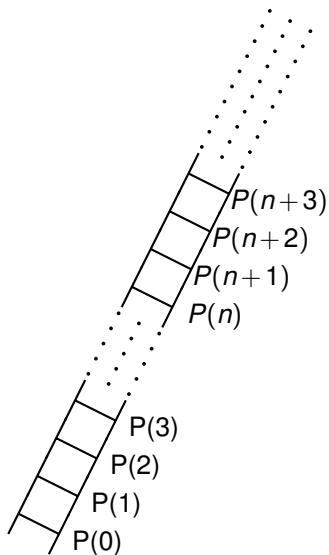
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Your favorite example of forever..

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Your favorite example of forever..or the natural numbers...

Induction

The canonical way of proving statements of the form

$$(\forall k \in \mathbf{N})(P(k))$$

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- ▶ For all $n \in \mathbb{N}$, $n^3 - n$ is divisible by 3.
- ▶ The sum of the first n odd integers is a perfect square.

The basic form

- ▶ Prove $P(0)$. “Base Case”.
- ▶ $P(k) \implies P(k+1)$
 - ▶ Assume $P(k)$, “Induction Hypothesis”
 - ▶ Prove $P(k+1)$. “Induction Step.”

$P(n)$ true for all natural numbers $n!!!$

Get to use $P(k)$ to prove $P(k+1)!!!$

Induction

The canonical way of proving statements of the form

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Gauss and Induction

Child Gauss: $(\forall n \in \mathbf{N})(\sum_{i=0}^n i = \frac{n(n+1)}{2})$

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Is predicate, $P(n)$ true for $n = k + 1$?

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Need to start somewhere. $P(0)$ is $\sum_{i=0}^0 i = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true

plus inductive step \implies true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$

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Predicate, $P(n)$, True for all natural numbers! **Proof by Induction.**

Poll: What did Gauss use in the proof?

- (A) Every natural number has a next number.
- (B) The recursive leap of faith.
- (C) $2^k > k$.
- (D) $\forall k \in \mathbb{N}, \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$.

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Theorem: For every $n \in \mathbb{N}$, $n^3 - n$ is divisible by 3. ($3 \mid (n^3 - n)$).

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Two color theorem: example.

Any map formed by dividing the plane into regions by drawing straight lines can be properly colored with two colors.

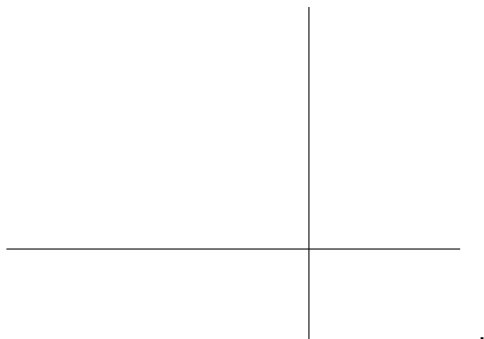


.

Proper coloring: for each line segment the regions on the two sides have different colors.¹

Two color theorem: example.

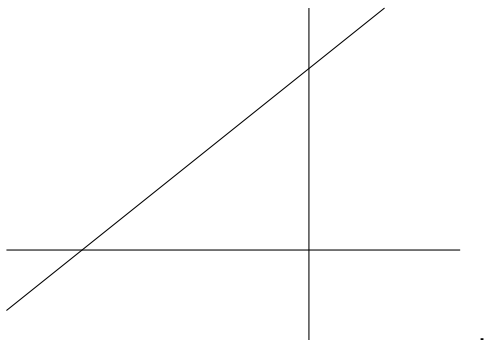
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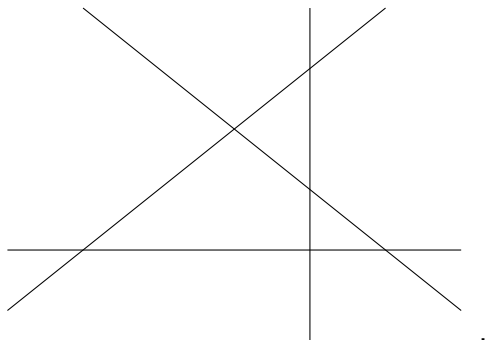
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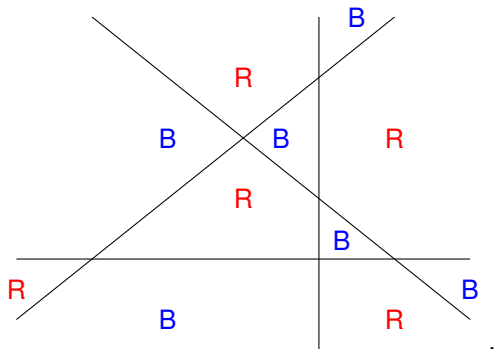
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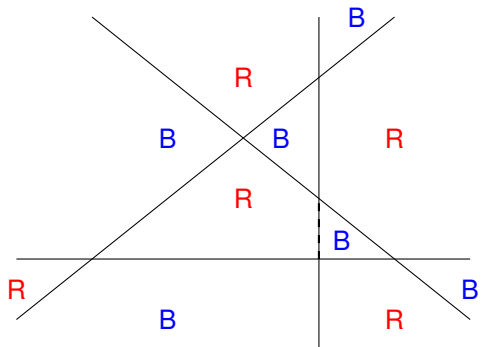
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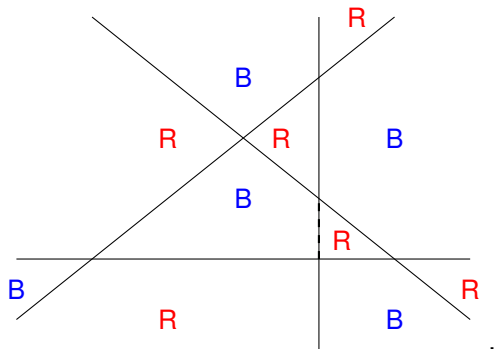


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Fact: Swapping red and blue gives another valid coloring.

Two color theorem: example.

Any map formed by dividing the plane into regions by drawing straight lines can be properly colored with two colors.



Proper coloring: for each line segment the regions on the two sides have different colors.¹

Fact: Swapping red and blue gives another valid coloring.

Two color theorem: proof illustration.

Base Case.

Two color theorem: proof illustration.

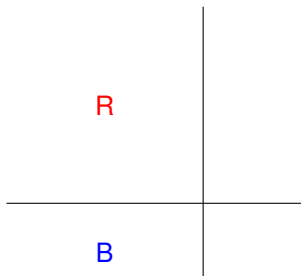
R



B

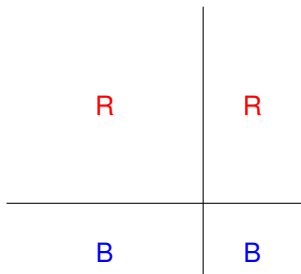
Base Case.

Two color theorem: proof illustration.



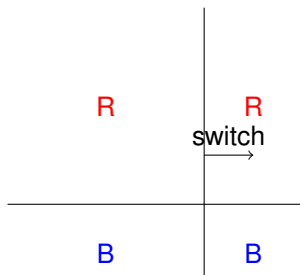
1. Add line.

Two color theorem: proof illustration.



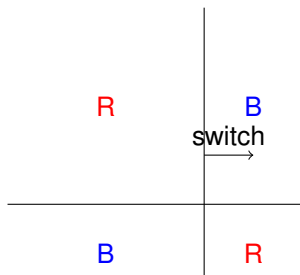
1. Add line.
2. Get inherited color for split regions

Two color theorem: proof illustration.



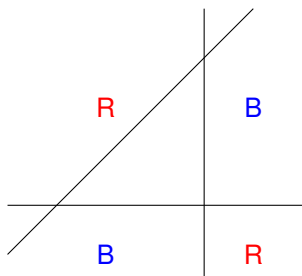
1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.
(Fixes conflicts along new line, and makes no new ones along previous line.)

Two color theorem: proof illustration.



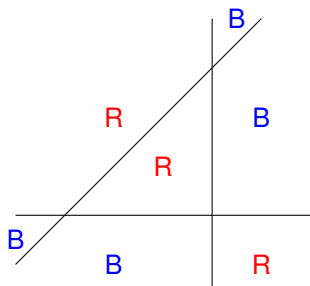
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Two color theorem: proof illustration.



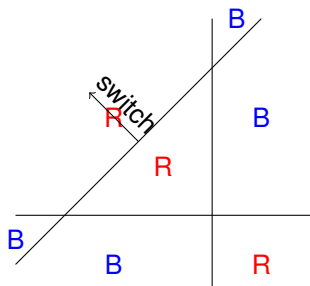
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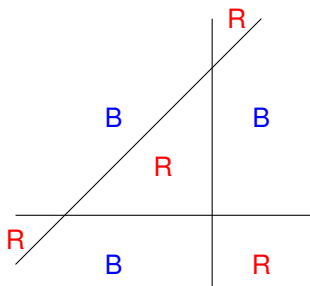
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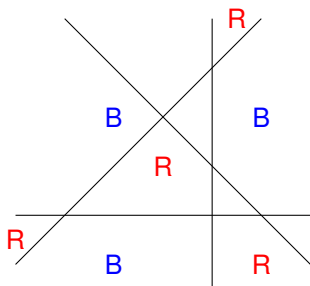
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Two color theorem: proof illustration.



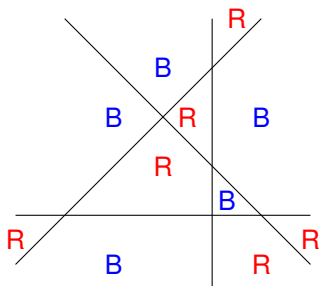
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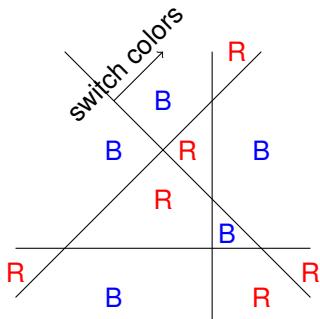
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Two color theorem: proof illustration.



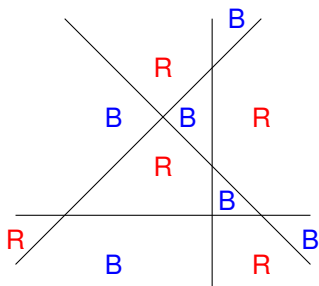
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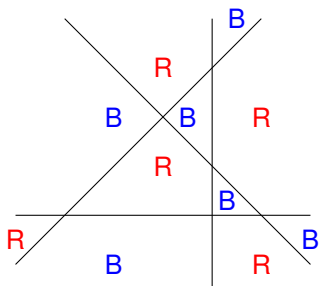
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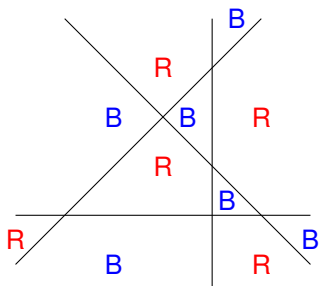
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Algorithm gives $P(k) \implies P(k+1)$.

Two color theorem: proof illustration.



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Algorithm gives $P(k) \implies P(k+1)$.



Poll: what did we use in the proof.

- (A) Switching a 2-coloring is a valid coloring.
- (B) Definition of 2-coloring.
- (C) Definition of adjacent.
- (D) Definition of region.
- (E) The four color theorem.

Strengthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.

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Base Case 1 (first odd number) is 1^2 .

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- Induction Step**
1. The $(k + 1)$ st odd number is $2k + 1$.
 2. Sum of the first $k + 1$ odds is $a^2 + 2k + 1$

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... P(k+1)!

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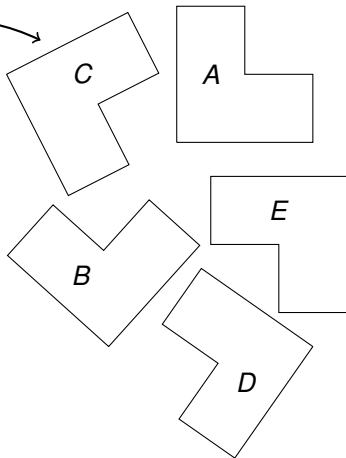
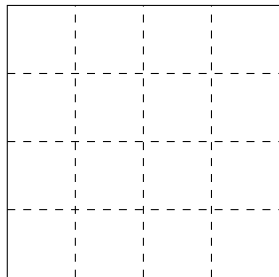
... P(k+1)!



Tiling Cory Hall Courtyard.

Use these *L*-tiles.

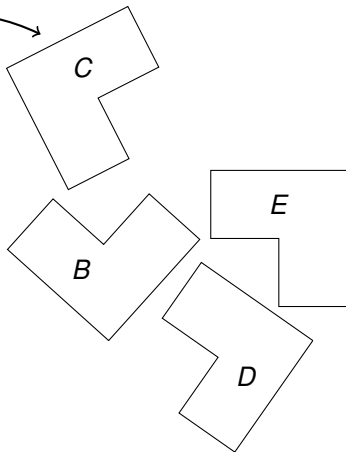
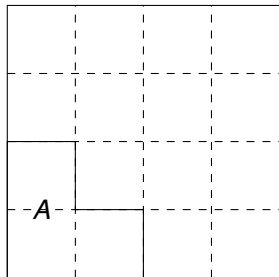
To Tile this 4×4 courtyard.



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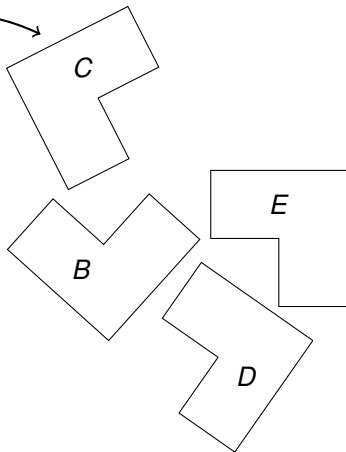
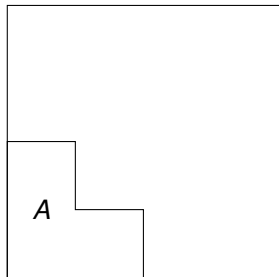
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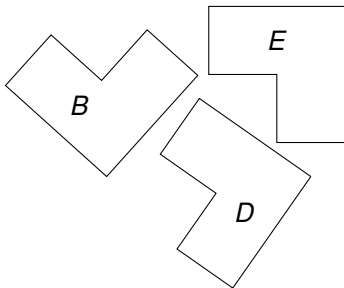
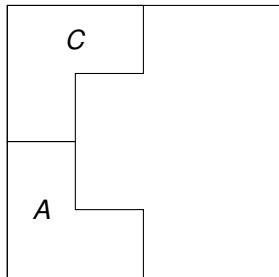
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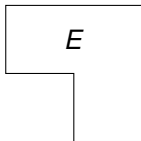
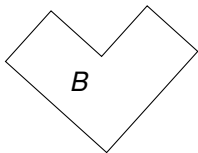
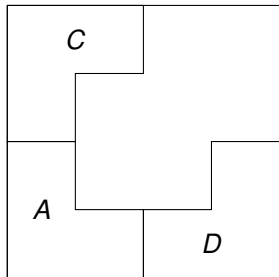
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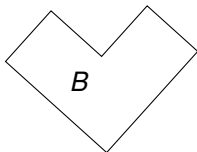
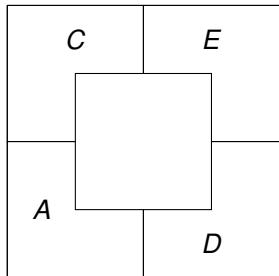
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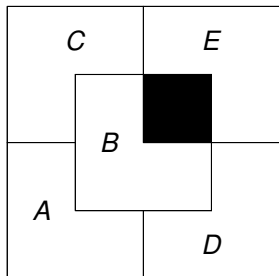
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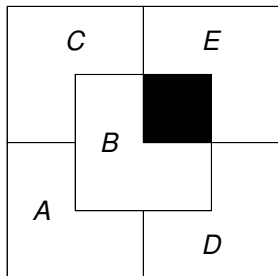
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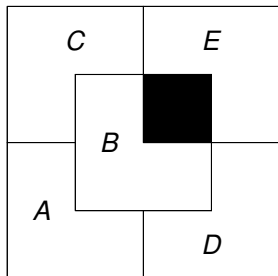


Alright!

Tiling Cory Hall Courtyard.

Use these L -tiles.

To Tile this 4×4 courtyard.

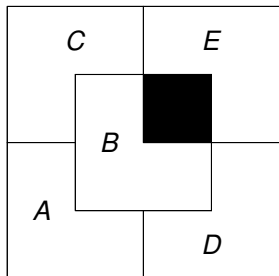


Alright!
Tiled 4×4 square with 2×2 L -tiles.

Tiling Cory Hall Courtyard.

Use these L -tiles.

To Tile this 4×4 courtyard.

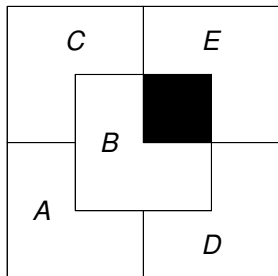


Alright!
Tiled 4×4 square with 2×2 L -tiles.
with a center hole.

Tiling Cory Hall Courtyard.

Use these L -tiles.

To Tile this 4×4 courtyard.



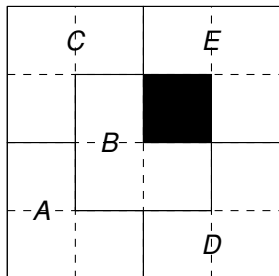
Alright!
Tiled 4×4 square with 2×2 L -tiles.
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Can we tile any $2^n \times 2^n$ with L -tiles (with a hole)

Tiling Cory Hall Courtyard.

Use these L -tiles.

To Tile this 4×4 courtyard.



Alright!
Tiled 4×4 square with 2×2 L -tiles.
with a center hole.

Can we tile any $2^n \times 2^n$ with L -tiles (with a hole) **for every n !**

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

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Base case: true for $k = 0$.

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$$2^{2(k+1)}$$

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$$2^{2(k+1)} = 2^{2k} * 2^2$$

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a integer $\implies (4a + 1)$ is an integer.

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Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

Proof:

Hole in center?

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Proof:

Base case: A single tile works fine.

Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

Proof:

Base case: A single tile works fine.

The hole is adjacent to the center of the 2×2 square.

Hole in center?

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Proof:

Base case: A single tile works fine.

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Induction Hypothesis:

Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

Proof:

Base case: A single tile works fine.

The hole is adjacent to the center of the 2×2 square.

Induction Hypothesis:

Any $2^n \times 2^n$ square can be tiled with a hole at the center.

Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

Proof:

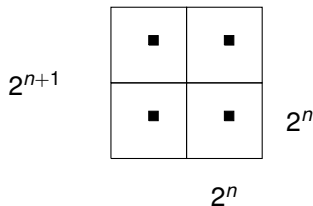
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Induction Hypothesis:

Any $2^n \times 2^n$ square can be tiled with a hole at the center.

$$2^{n+1}$$



Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

Proof:

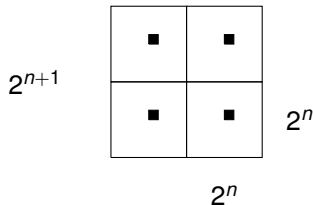
Base case: A single tile works fine.

The hole is adjacent to the center of the 2×2 square.

Induction Hypothesis:

Any $2^n \times 2^n$ square can be tiled with a hole at the center.

$$2^{n+1}$$



What to do now???

Hole can be anywhere!

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“Any $2^n \times 2^n$ square can be tiled with a hole **anywhere**.”

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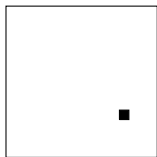


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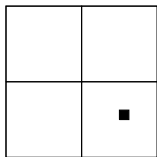


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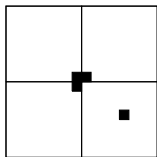


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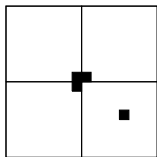


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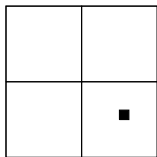


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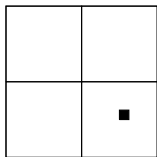


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For example. Use reduced form: a/b and order by $a+b$.

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Tournaments have short cycles

Def: A **round robin tournament on n players**: every player p plays every other player q , and either $p \rightarrow q$ (p beats q) or $q \rightarrow p$ (q beats p .)

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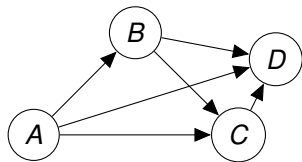
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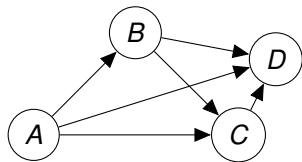
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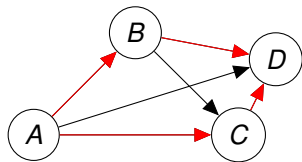


Theorem: Any tournament that has a cycle has a cycle of length 3.

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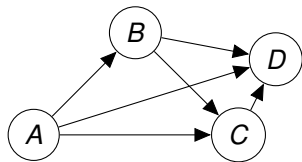


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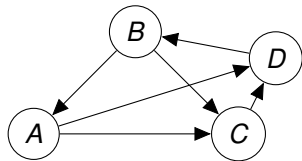


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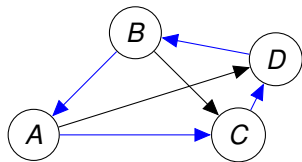


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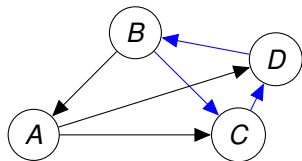


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Assume the the **smallest cycle** is of length k .

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Case 1: Of length 3.

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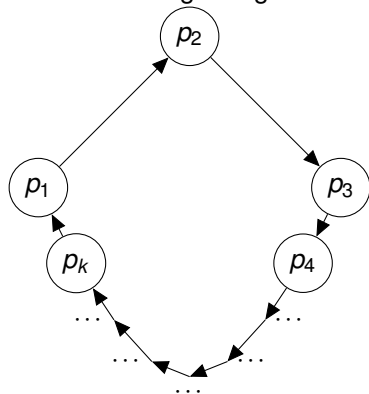
Case 2: Of length larger than 3.

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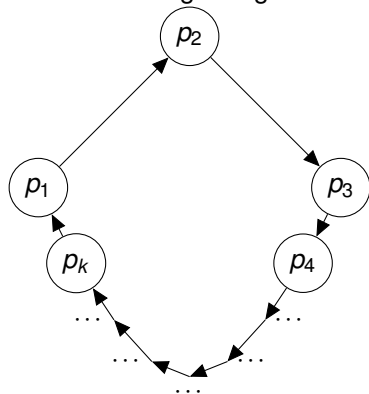


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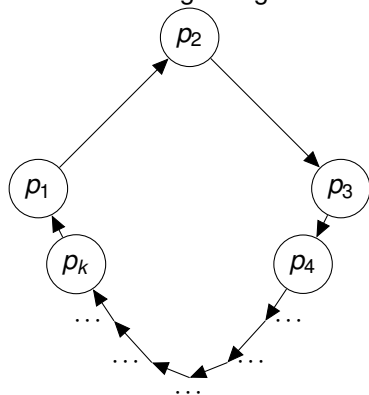


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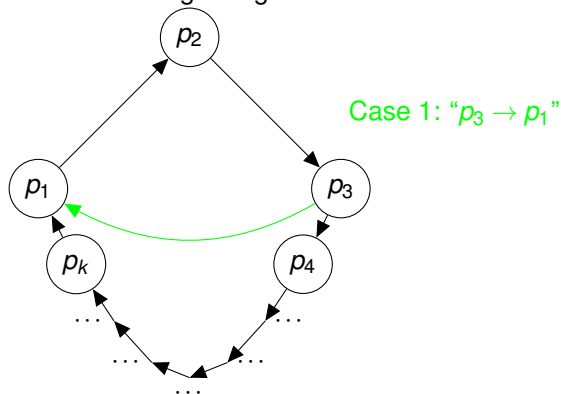


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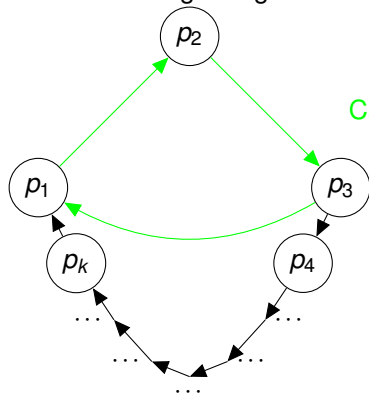


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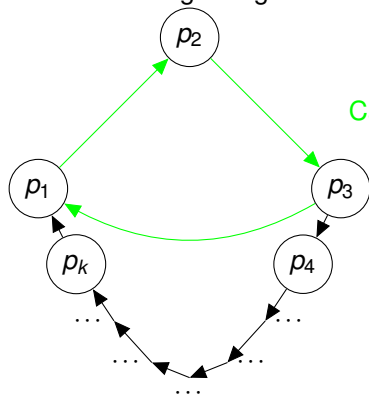
Case 1: " $p_3 \rightarrow p_1$ " \implies 3 cycle

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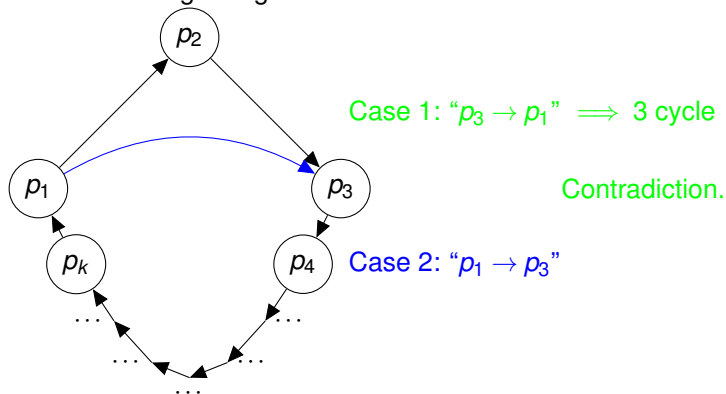
Contradiction.

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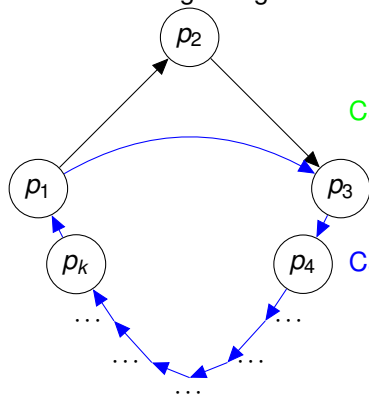


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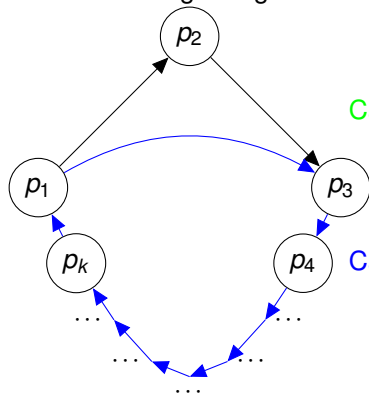
Case 2: " $p_1 \rightarrow p_3$ " $\implies k-1$ length cycle!

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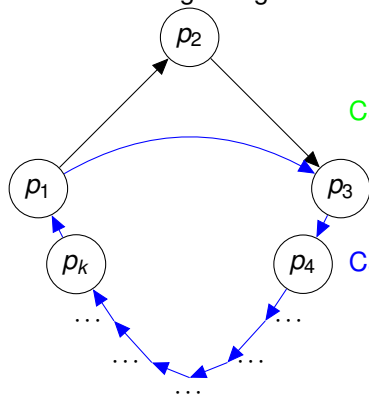
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Tournament on $n+1$ people,

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Tournament on $n+1$ people,
Remove arbitrary person

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Remove arbitrary person \rightarrow yield tournament on $n-1$ people.

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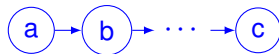


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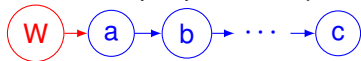


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Second k have same color by $P(k)$.

A horse in the middle in common!

Fix base case.

There are two horses of the same color. ...Still doesn't work!!

Horses of the same color...

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.

New Base Case: $P(2)$: there are two horses with same color.

Induction Hypothesis: $P(k)$ - Any k horses have the same color.

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Of course it doesn't work.

As we will see, it is more subtle to catch errors in proofs of correct theorems!!

Sad Islanders...

Island with 100 possibly blue-eyed and green-eyed inhabitants.

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Why?

They know induction.

Thm: If there are n villagers with green eyes they do ritual on day n .

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On day $n + 1$, a green eyed person sees n people with green eyes.

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Wait! Visitor added no information.

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

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Another example:

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No one knows other people see that he has no clothes.

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Another example:

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No one knows other people see that he has no clothes.

Until kid points it out.

Summary: principle of induction.

Today: More induction.

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$(P(0))$

Summary: principle of induction.

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$$(P(0) \wedge ((\forall k \in \mathbb{N})(P(k) \implies P(k+1))))$$

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Hole anywhere.

Not same as strong induction. E.g., used in product of primes proof.

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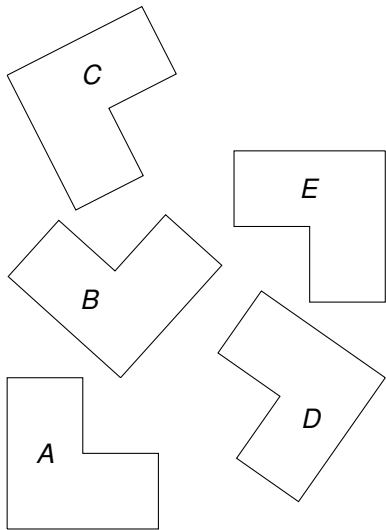
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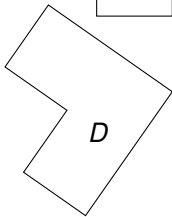
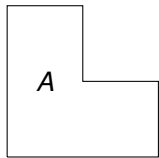
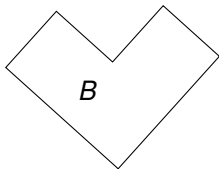
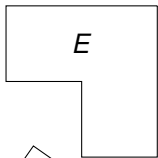
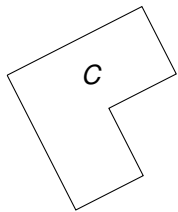
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Induction \equiv Recursion.

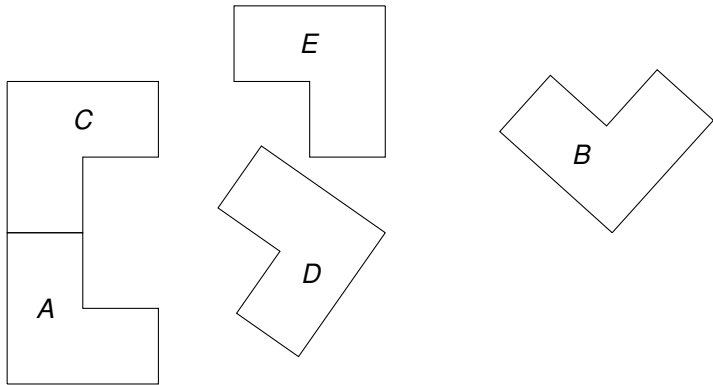
Tiling Cory Hall Courtyard.



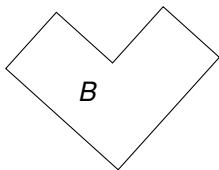
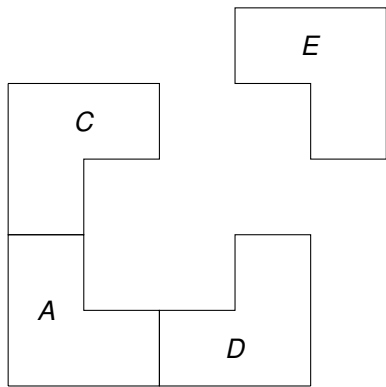
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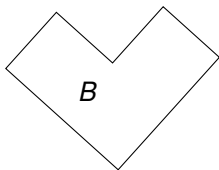
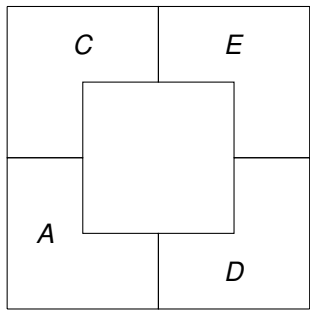
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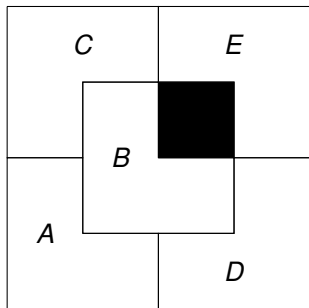
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