Outline

Linear Regression: wrapup.

How do I love e?

Balls in Bins.

- Birthday.
- Coupon Collector.
- Load balancing.

Poisson Distribution: Sum of two Poissons is Poisson.

Estimation Error

We saw that the LLSE of Y given X is

$$L[Y|X] = \hat{Y} = E[Y] + \frac{cov(X,Y)}{var(X)}(X - E[X]).$$

How good is this estimator?
Or what is the mean squared estimation error?
We find

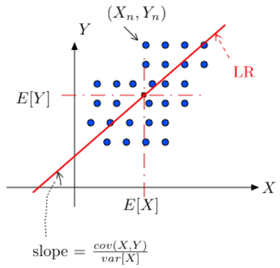
$$\begin{aligned}
E[|Y - L[Y|X]|^2] &= E[(Y - E[Y] - (cov(X,Y)/var(X))(X - E[X]))^2] \\
&= E[(Y - E[Y])^2] - 2\frac{cov(X,Y)}{var(X)}E[(Y - E[Y])(X - E[X])] \\
&\quad + (\frac{cov(X,Y)}{var(X)})^2 E[(X - E[X])^2] \\
&= var(Y) - \frac{cov(X,Y)^2}{var(X)}.
\end{aligned}$$

Without observations, the estimate is $E[Y]$. The error is $var(Y)$. Observing X reduces the error.

Dividing by $var(Y)$, one gets reduction: $\frac{(cov(X,Y))^2}{var(Y)var(Y)} = (corr(X,Y))^2 = r^2$.

LR: Another Figure

$$L[Y|X] = \hat{Y} = E[Y] + \frac{cov(X,Y)}{var(X)}(X - E[X]).$$



Note that

- the LR line goes through $(E[X], E[Y])$
- its slope is $\frac{cov(X,Y)}{var(X)}$.

Quadratic Regression

Let X, Y be two random variables defined on the same probability space.

Definition: The quadratic regression of Y over X is the random variable

$$Q[Y|X] = a + bX + cX^2$$

where a, b, c are chosen to minimize $E[(Y - a - bX - cX^2)^2]$.

Derivation: We set to zero the derivatives w.r.t. a, b, c. We get

$$\begin{aligned}
0 &= E[Y - a - bX - cX^2] = E[Y] - a - bE[X] - cE[X^2] \\
0 &= E[(Y - a - bX - cX^2)X] = E[XY] - a - bE[X^2] - cE[X^3] \\
0 &= E[(Y - a - bX - cX^2)X^2] = E[X^2Y] - aE[X^2] - bE[X^3] - cE[X^4]
\end{aligned}$$

We solve these three equations in the three unknowns (a, b, c).

For linear regression, $L[Y|X]$, approach gives:

$$L[Y|X] = \hat{Y} = E[Y] + \frac{cov(X,Y)}{var(X)}(X - E[X]).$$

How do I love e?

Let me count the ways.

What is e?
For a function $f(x) = e^x$, $f'(x) = e^x$.
Another view: $\frac{dy}{dx} = y$.
More money you have the faster you gain money.
More rabbits there are, the more rabbits you get.
More people with a disease the faster it grows:
Epidemiologists:reproduction rate, R.
Discrete version: $x_{n+1} - x_n = \Delta(x_n) = x_n$.
$x_n = 2^n$, for $x_0 = 1$.

How do I love e?

For a function $f(x) = e^x$, $f'(x) = e^x$.

What is this $f'(x)$?
Slope of the tangent line.

$$f'(x) \approx \frac{f(x + 1/n) - f(x)}{x + 1/n - x} = \frac{f(x + 1/n) - f(x)}{1/n}$$

for large n.

And $f(x) = e^x$, $f(x + 1/n) = e^{x + 1/n} = e^x e^{1/n}$, so

$$f'(x) \approx \frac{e^x(e^{1/n} - 1)}{1/n} = e^x \frac{e^{1/n} - 1}{1/n} \approx e^x$$

$$\implies \frac{e^{1/n} - 1}{1/n} \approx 1 \implies e^{1/n} = 1/n \implies e \approx (1 + 1/n)^n.$$

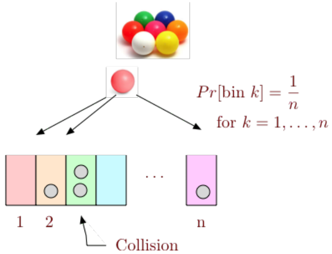
Continuous compounded interest: rate r.
break time into intervals of size $1/n$.
$(1 + r/n)^n \to ((1 + r/n)^{n/r})^r \to e^r$.

Balls in bins

One throws m balls into $n > m$ bins.

Balls in bins

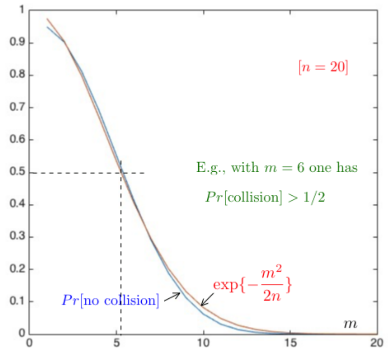
One throws m balls into $n > m$ bins.



Theorem:
$Pr[\text{no collision}] \approx \exp\{-\frac{m^2}{2n}\}$, for large enough n.

Balls in bins

Theorem:
$Pr[\text{no collision}] \approx \exp\{-\frac{m^2}{2n}\}$, for large enough n.



Balls in bins

Theorem:
$Pr[\text{no collision}] \approx \exp\{-\frac{m^2}{2n}\}$, for large enough n.

In particular, $Pr[\text{no collision}] \approx 1/2$ for $m^2/(2n) \approx \ln(2)$, i.e.,

$$m \approx \sqrt{2\ln(2)n} \approx 1.2\sqrt{n}.$$

E.g., $1.2\sqrt{20} \approx 5.4$.

Roughly, $Pr[\text{collision}] \approx 1/2$ for $m = \sqrt{n}$. ($e^{-0.5} \approx 0.6$.)

The Calculation.

A_i = no collision when ith ball is placed in a bin.

$Pr[A_i | A_{i-1} \cap \cdots \cap A_1] = (1 - \frac{i-1}{n})$.

no collision = $A_1 \cap \cdots \cap A_m$.

Product rule:

$Pr[A_1 \cap \cdots \cap A_m] = Pr[A_1]Pr[A_2|A_1] \cdots Pr[A_m|A_1 \cap \cdots \cap A_{m-1}]$

$$\Rightarrow Pr[\text{no collision}] = \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{m-1}{n}\right).$$

Hence,

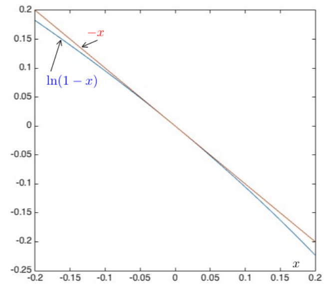
$$\ln(Pr[\text{no collision}]) = \sum_{k=1}^{m-1} \ln(1 - \frac{k}{n}) \approx \sum_{k=1}^{m-1} (-\frac{k}{n})^{(*)}$$

$$= -\frac{1}{n}\frac{m(m-1)}{2}^{(\dagger)} \approx -\frac{m^2}{2n}$$

(*) We used $\ln(1 - \varepsilon) \approx -\varepsilon$ for $|\varepsilon| \ll 1$.

(†) $1 + 2 + \cdots + m - 1 = (m-1)m/2$.

Approximation



$$\exp\{-x\} = 1 - x + \frac{1}{2!}x^2 + \cdots \approx 1 - x, \text{ for } |x| \ll 1.$$

Hence, $-x \approx \ln(1 - x)$ for $|x| \ll 1$.

Today's your birthday, it's my birthday too..

Probability that m people all have different birthdays?
With $n = 365$, one finds

Pr[collision] $\approx 1/2$ if $m \approx 1.2\sqrt{365} \approx 23$.

If $m = 60$, we find that

$$Pr[\text{no collision}] \approx \exp\{-\frac{m^2}{2n}\} = \exp\{-\frac{60^2}{2 \times 365}\} \approx 0.007.$$

If $m = 366$, then Pr[no collision] $= 0$. (No approximation here!)

Using linearity of expectation.

Experiment: m balls into n bins uniformly at random.

Random Variable:
X = Number of collisions between pairs of balls.

or number of pairs i and j where ball i and ball j are in same bin.

$X_{ij} = 1\{\text{balls } i,j \text{ in same bin}\}$

$X = \sum_{ij} X_{ij}$

$E[X_{ij}] = Pr[\text{balls } i,j \text{ in same bin}] = \frac{1}{n}$.

Ball i in some bin, ball j chooses that bin with probability $1/n$.

$E[X] = \frac{m(m-1)}{2n} \approx \frac{m^2}{2n}$.

For $m = \sqrt{n}$, $E[X] = 1/2$

Markov: $Pr[X \geq c] \leq \frac{EX}{c}$.

$Pr[X \geq 1] \leq \frac{E[X]}{1} = 1/2$.

Checksums!

Consider a set of m files.
Each file has a checksum of b bits.
How large should b be for Pr[share a checksum] $\leq 10^{-3}$?

Claim: $b \geq 2.9\ln(m) + 9$.

Proof:

Let $n = 2^b$ be the number of checksums.
We know $Pr[\text{no collision}] \approx \exp\{-m^2/(2n)\} \approx 1 - m^2/(2n)$. Hence,

$$Pr[\text{no collision}] \approx 1 - 10^{-3} \Leftrightarrow m^2/(2n) \approx 10^{-3}$$
$$\Leftrightarrow 2n \approx m^2 10^3 \Leftrightarrow 2^{b+1} \approx m^2 2^{10}$$
$$\Leftrightarrow b + 1 \approx 10 + 2\log_2(m) \approx 10 + 2.9\ln(m).$$

Note: $\log_2(x) = \log_2(e)\ln(x) \approx 1.44\ln(x)$.

Coupon Collector Problem.

There are n different baseball cards.
(Brian Wilson, Jackie Robinson, Roger Hornsby, ...)

One random baseball card in each cereal box.

Theorem: If you buy m boxes,

(a) Pr[miss one specific item] $\approx e^{-\frac{m}{n}}$

(b) Pr[miss any one of the items] $\leq ne^{-\frac{m}{n}}$.

Coupon Collector Problem: Analysis.

Event A_m = 'fail to get Brian Wilson in m cereal boxes'

Fail the first time: $(1 - \frac{1}{n})$
Fail the second time: $(1 - \frac{1}{n})$
And so on ... for m times. Hence,

$$\begin{aligned} Pr[A_m] &= (1-\frac{1}{n}) \times \cdots \times (1-\frac{1}{n}) \\ &= (1-\frac{1}{n})^m \\ ln(Pr[A_m]) &= m\ln(1-\frac{1}{n}) \approx m \times (-\frac{1}{n}) \\ Pr[A_m] &\approx \exp\{-\frac{m}{n}\}. \end{aligned}$$

For $p_m = \frac{1}{2}$, we need around $n\ln 2 \approx 0.69n$ boxes.

Collect all cards?

Experiment: Choose m cards at random with replacement.

Events: E_k = 'fail to get player k' , for k = 1, ..., n

Probability of failing to get at least one of these n players:

$$p := Pr[E_1 \cup E_2 \cdots \cup E_n]$$

How does one estimate p? Union Bound:

$$p = Pr[E_1 \cup E_2 \cdots \cup E_n] \leq Pr[E_1] + Pr[E_2] \cdots Pr[E_n].$$

$$Pr[E_k] \approx e^{-\frac{m}{n}}, k = 1, \ldots, n.$$

Plug in and get

$$p \leq ne^{-\frac{m}{n}}.$$

Collect all cards?

Thus,

$$Pr[\text{missing at least one card}] \le ne^{-\frac{m}{n}}.$$

Hence,

$$Pr[\text{missing at least one card}] \le p \text{ when } m \ge n\ln(\frac{n}{p}).$$

To get $p = 1/2$, set $m = n\ln(2n)$.

$(p \le ne^{-\frac{m}{n}} \le ne^{-\ln(n/p)} \le n(\frac{p}{n}) \le p.)$

E.g., $n = 10^2 \Rightarrow m = 530; n = 10^3 \Rightarrow m = 7600$.

Time to collect coupons

X-time to get n coupons.

X_1 - time to get first coupon. Note: $X_1 = 1$. $E(X_1) = 1$.

X_2 - time to get second coupon after getting first.

Pr[“get second coupon”|“got ~~milk~~ first coupon”] $= \frac{n-1}{n}$

$E[X_2]$? Geometric ! ! ! $\implies E[X_2] = \frac{1}{p} = \frac{1}{\frac{n-1}{n}} = \frac{n}{n-1}$.

Pr[“getting ith coupon|“got $i-1$rst coupons”] $= \frac{n-(i-1)}{n} = \frac{n-i+1}{n}$

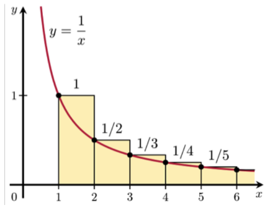
$E[X_i] = \frac{1}{p} = \frac{n}{n-i+1}, i = 1,2,\ldots,n$.

$$\begin{aligned} E[X] &= E[X_1] + \cdots + E[X_n] = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1} \\ &= n(1 + \frac{1}{2} + \cdots + \frac{1}{n}) =: nH(n) \approx n(\ln n + \gamma) \end{aligned}$$

Review: Harmonic sum

$$H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \int_1^n \frac{1}{x}dx = \ln(n).$$

.



A good approximation is

$H(n) \approx \ln(n) + \gamma$ where $\gamma \approx 0.58$ (Euler-Mascheroni constant).

Simplest..

Load balance: m balls in n bins.

For simplicity: n balls in n bins.

Round robin: load 1 !

Centralized! Not so good.

Uniformly at random? Average load 1.

Max load?

n. Uh Oh!

Max load with probability $\ge 1-\delta$?

$\delta = \frac{1}{n^c}$ for today. c is 1 or 2.

Balls in bins.

For each of n balls, choose random bin: X_i balls in bin i.

$Pr[X_i \ge k] \le \sum_{S \subseteq [n], |S| = k} Pr[\text{balls in } S \text{ chooses bin } i]$

From Union Bound: $Pr[\cup_i A_i] \le \sum_i Pr[A_i]$

$Pr[\text{balls in } S \text{ chooses bin } i] = (\frac{1}{n})^k$ and $\binom{n}{k}$ subsets S.

$$\begin{aligned} \Pr[X_i \ge k] &\le \binom{n}{k}\left(\frac{1}{n}\right)^k \\ &\le \frac{n^k}{k!}\left(\frac{1}{n}\right)^k = \frac{1}{k!} \end{aligned}$$

Choose k, so that $Pr[X_i \ge k] \le \frac{1}{n^2}$.

$Pr[\text{any } X_i \ge k] \le n \times \frac{1}{n^2} = \frac{1}{n} \to$ max load $\le k$ w.p. $\ge 1 - \frac{1}{n}$

Solving for k

$Pr[X_i \ge k] \le \frac{1}{k!} \le 1/n^2$?

What is upper bound on max-load k?

Lemma: Max load is $\Theta(\log n)$ with probability $\ge 1 - \frac{1}{n}$.

$k! \ge n^2$ for $k = 2e\log n$

(Recall $k! \ge (\frac{k}{e})^k$.)

$\implies \frac{1}{k!} \le (\frac{e}{k})^k \le \left(\frac{1}{2\log n}\right)^k$

If $\log n \ge 1$, then $k = 2e\log n$ suffices.

Also: $k = \Theta(\log n / \log\log n)$ suffices as well.

$k^k \to n^c$.

Actually Max load is $\Theta(\log n/\log\log n)$ w.h.p.

(W.h.p. - means with probability at least $1 - O(1/n^c)$ for today.)

Better than variance based methods...

Sum of Poisson Random Variables.

For $X = P(\lambda)$, $Pr[X=i] = e^{-\lambda}\frac{\lambda^i}{i!}$

For $X = P(\lambda)$ and $Y = P(\mu)$, what is distribution $X+Y$?

$Pr[X+Y=k] = e^{-\lambda-\mu}\sum_{i+j=k}\frac{\lambda^i\mu^j}{i!j!}$.

Poission? Yes.
What parameter? $\lambda+\mu$.

Why?
$P(\lambda)$ is limit $n\to\infty$ of $B(n,\lambda/n)$.

Recall Derivation:
break interval into n intervals
and each has arrival with probability λ/n.

Now:
arrival for X happens with probability λ/n
arrival for Y happens with probability μ/n

So, we get limit $n\to\infty$ is $B(n,(\lambda+\mu)/n)$.

Details: both could arrive with probability $\lambda\mu/n^2$.
But this goes to zero as $n\to\infty$.
(Like λ^2/n^2 in previous derivation)

Discrete Probability.

Probability Space: Ω, $Pr : \Omega \to [0,1]$, $\sum_{\omega\in\Omega} Pr(w) = 1$.

Events: $A \subset \Omega$, $Pr[A] = \sum_{\omega\in A} Pr[\omega]$.

$Pr[A\cup B] = Pr[A] + Pr[B] - Pr[A\cap B]$

Simple Total Probability: $Pr[B] = Pr[A\cap B] + Pr[\overline{A}\cap B]$.

Conditional Probability: $Pr[A|B] = \frac{Pr[A\cap B]}{Pr[B]}$.

Simple Product Rule: $Pr[A\cap B] = Pr[A|B]Pr[B]$.

Bayes Rule: $Pr[A|B] = \frac{Pr[B|A]Pr[B]}{Pr[B]}$

Inference:
Have one of two coins. Flip coin, which coin do you have?
Got positive test result. What is probability you have disease?

Random Variables

Random Variables: $X : \Omega \to R$.

Distribution: $Pr[X=a] = \sum_{\omega : X(\omega)=a} Pr(\omega)$

X and Y independent $\iff$ all associated events are independent.
Expectation: $E[X] = \sum_a aPr[X=a] = \sum_{\omega\in\Omega} X(\omega)Pr(\omega)$.
Linearity: $E[X+Y] = E[X] + E[Y]$.

Variance: $Var(X) = E[(X-E[X])^2] = E[X^2] - (E(X))^2$
For independent X,Y, $Var(X+Y) = Var(X) + Var(Y)$.
Also: $Var(cX) = c^2 Var(X)$ and $Var(X+b) = Var(X)$.

Poisson: $X \sim P(\lambda)$ $\quad Pr[X=i] = e^{-\lambda}\frac{\lambda^i}{i!}$.
$E(X) = \lambda$, $Var(X) = \lambda$.
Binomial: $X \sim B(n,p)$ $\quad Pr[X=i] = \binom{n}{i}p^i(1-p)^{n-i}$
$E(X) = np$, $Var(X) = np(1-p)$
Uniform: $X \sim U\{1,\ldots,n\}$ $\quad \forall i \in [1,n], Pr[X=i] = \frac{1}{n}$.
$E[X] = \frac{n+1}{2}$, $Var(X) = \frac{n^2-1}{12}$.
Geometric: $X \sim G(p)$ $\quad Pr[X=i] = (1-p)^{i-1}p$
$E(X) = \frac{1}{p}$, $Var(X) = \frac{1-p}{p^2}$

Note: Probability Mass Function $\equiv$ Distribution.

Concentration: Law Of Large Numbers.

Markov: For a non-negative r.v. X, $Pr[X\geq c] \leq \frac{E[X]}{c}$.

Chebyshev: For a random variable X: $Pr[|X-E(X)|>\varepsilon] \leq \frac{Var(X)}{epsilon^2}$

For $X = \frac{X_1+\cdots+X_n}{n}$, where X_i are indentical and independent.
$Var(X) = \frac{var(X_i)}{n}$.

Law of Large Numbers: $A_n = \frac{X_1+\cdots+X_n}{n}$.
$\lim_{n\to} A_n = E[X_1]$.
Cuz:
$Pr[|A_n - E[A_n]| \geq \varepsilon] \leq \frac{var A_n}{\varepsilon^2} = \frac{var(X_1)}{n\varepsilon^2}$.

For X_i with $Var(X_i) = \sigma^2$.
What is the confidence interval for A_n for confidence .95?
For what ε is $Pr[|A_n - E[A_n]| \geq \varepsilon] \leq .05 = \delta$?
$\varepsilon = \frac{\sigma}{\sqrt{n\delta}}$ using Chebyshev.
$\varepsilon \approx \frac{\sigma}{\sqrt{n}}\log\frac{1}{\delta}$ using "Chernoff."
"z-score" from AP statistics.
FYI: Chebyshev uses $E[X^2]$, Chernoff uses $E[e^X]$. Both use Markov.

Joint Distributions and Estimation.

Distribution for X,Y: $Pr[X=a, Y=b]$.
Marginals: $Pr[X=a] = \sum_b Pr[X=a, Y=b]$.

Conditioning:
$Pr[X=a|Y=b] = \frac{Pr[X=a,Y=b]}{Pr[Y=b]}$
$E[Y|X] = \sum_b b \times Pr[Y=b|X]$.

Estimation minimizing Mean Squared Error:
$E[X]$ for X. Error is $var(X)$.
$E[Y|X]$ for Y if you know X.
Best linear function.
$L[Y|X] = E[Y] + corr(X,Y)\sqrt{var(Y)}\frac{X-E(X)}{\sqrt{var(X)}}$.
Reduces mean squared error Y by $(corr(X,Y))^2$ by $var(Y)$.

Warning: assume knowing joint distribution.
Statistics: sampling....Law of Large Numbers.
Computer Science: large data, other functions "Deep Networks."