Random Variables

Random Variables: $X : \Omega \rightarrow R$.

Random Variables

Random Variables: $X : \Omega \rightarrow R$.

Random Variables

Random Variables: $X : \Omega \to R$.

Distribution: $Pr[X = a] = \sum_{\omega : X(\omega) = a} Pr(\omega)$

Random Variables

Random Variables: $X : \Omega \to R$.

Distribution: $Pr[X = a] = \sum_{\omega : X(\omega) = a} Pr(\omega)$

Random Variables

Random Variables: $X : \Omega \to R$.

Distribution: $Pr[X = a] = \sum_{\omega : X(\omega) = a} Pr(\omega)$

X and Y independent $\iff$ all associated events are independent.

Random Variables

Random Variables: $X : \Omega \to R$.

Distribution: $Pr[X = a] = \sum_{\omega : X(\omega) = a} Pr(\omega)$

X and Y independent $\iff$ all associated events are independent.
Expectation: $E[X] = \sum_a a Pr[X = a] = \sum_{\omega \in \Omega} X(\omega) Pr(\omega)$.

Random Variables

Random Variables: $X : \Omega \to R$.

Distribution: $Pr[X = a] = \sum_{\omega : X(\omega) = a} Pr(\omega)$

X and Y independent $\iff$ all associated events are independent.
Expectation: $E[X] = \sum_a aPr[X = a] = \sum_{\omega \in \Omega} X(\omega) Pr(\omega)$.
  Linearity: $E[X + Y] = E[X] + E[Y]$.

Random Variables

Random Variables: $X : \Omega \to R$.

Distribution: $Pr[X = a] = \sum_{\omega : X(\omega)=a} Pr(\omega)$

X and Y independent $\iff$ all associated events are independent.
Expectation: $E[X] = \sum_a aPr[X = a] = \sum_{\omega \in \Omega} X(\omega)Pr(\omega)$.
Linearity: $E[X + Y] = E[X] + E[Y]$.

Variance: $Var(X) = E[(X - E[X])^2] = E[X^2] - (E(X))^2$

Random Variables

Random Variables: $X : \Omega \to R$.

Distribution: $Pr[X = a] = \sum_{\omega : X(\omega)=a} Pr(\omega)$

X and Y independent $\iff$ all associated events are independent.
Expectation: $E[X] = \sum_a aPr[X = a] = \sum_{\omega \in \Omega} X(\omega)Pr(\omega)$.
Linearity: $E[X+Y] = E[X]+E[Y]$.

Variance: $Var(X) = E[(X-E[X])^2] = E[X^2]-(E(X))^2$
For independent X, Y, $Var(X+Y) = Var(X)+Var(Y)$.

Random Variables

Random Variables: $X : \Omega \to R$.

Distribution: $Pr[X = a] = \sum_{\omega : X(\omega) = a} Pr(\omega)$

X and Y independent $\iff$ all associated events are independent.
Expectation: $E[X] = \sum_a aPr[X = a] = \sum_{\omega \in \Omega} X(\omega) Pr(\omega)$.
  Linearity: $E[X + Y] = E[X] + E[Y]$.

Variance: $Var(X) = E[(X - E[X])^2] = E[X^2] - (E(X))^2$
  For independent X, Y, $Var(X + Y) = Var(X) + Var(Y)$.
  Also: $Var(cX) = c^2 Var(X)$ and $Var(X + b) = Var(X)$.

Random Variables

Random Variables: $X : \Omega \to R$.

Distribution: $Pr[X = a] = \sum_{\omega : X(\omega)=a} Pr(\omega)$

X and Y independent $\iff$ all associated events are independent.
Expectation: $E[X] = \sum_a aPr[X = a] = \sum_{\omega \in \Omega} X(\omega)Pr(\omega)$.
Linearity: $E[X + Y] = E[X] + E[Y]$.

Variance: $Var(X) = E[(X - E[X])^2] = E[X^2] - (E(X))^2$
For independent X, Y, $Var(X + Y) = Var(X) + Var(Y)$.
Also: $Var(cX) = c^2 Var(X)$ and $Var(X + b) = Var(X)$.

Poisson: $X \sim P(\lambda)$ $\quad Pr[X = i] = e^{-\lambda} \frac{\lambda^i}{i!}$.

Random Variables

Random Variables: $X : \Omega \to R$.

Distribution: $Pr[X = a] = \sum_{\omega : X(\omega) = a} Pr(\omega)$

X and Y independent $\iff$ all associated events are independent.
Expectation: $E[X] = \sum_a aPr[X = a] = \sum_{\omega \in \Omega} X(\omega)Pr(\omega)$.
  Linearity: $E[X + Y] = E[X] + E[Y]$.

Variance: $Var(X) = E[(X - E[X])^2] = E[X^2] - (E(X))^2$
  For independent X, Y, $Var(X + Y) = Var(X) + Var(Y)$.
  Also: $Var(cX) = c^2 Var(X)$ and $Var(X + b) = Var(X)$.

Poisson: $X \sim P(\lambda)$ $Pr[X = i] = e^{-\lambda} \frac{\lambda^i}{i!}$.
  $E(X) = \lambda$, $Var(X) = \lambda$.

Random Variables

Random Variables: $X:\Omega \rightarrow R$.

Distribution: $Pr[X=a] = \sum_{\omega:X(\omega)=a} Pr(\omega)$

X and Y independent $\iff$ all associated events are independent.
Expectation: $E[X] = \sum_a aPr[X=a] = \sum_{\omega\in\Omega} X(\omega)Pr(\omega)$.
Linearity: $E[X+Y] = E[X]+E[Y]$.

Variance: $Var(X) = E[(X-E[X])^2] = E[X^2]-(E(X))^2$
For independent X,Y, $Var(X+Y) = Var(X)+Var(Y)$.
Also: $Var(cX) = c^2 Var(X)$ and $Var(X+b) = Var(X)$.

Poisson: $X \sim P(\lambda)$ $\quad Pr[X=i] = e^{-\lambda}\frac{\lambda^i}{i!}$.
$E(X) = \lambda$, $Var(X) = \lambda$.
Binomial: $X \sim B(n,p)$

Random Variables

Random Variables: $X : \Omega \to R$.

Distribution: $Pr[X = a] = \sum_{\omega : X(\omega) = a} Pr(\omega)$

X and Y independent $\iff$ all associated events are independent.
Expectation: $E[X] = \sum_a aPr[X = a] = \sum_{\omega \in \Omega} X(\omega) Pr(\omega)$.
Linearity: $E[X + Y] = E[X] + E[Y]$.

Variance: $Var(X) = E[(X - E[X])^2] = E[X^2] - (E(X))^2$
For independent X, Y, $Var(X + Y) = Var(X) + Var(Y)$.
Also: $Var(cX) = c^2 Var(X)$ and $Var(X + b) = Var(X)$.

Poisson: $X \sim P(\lambda)$ $\quad Pr[X = i] = e^{-\lambda} \frac{\lambda^i}{i!}$.
$E(X) = \lambda$, $Var(X) = \lambda$.
Binomial: $X \sim B(n, p)$ $\quad Pr[X = i] = \binom{n}{i} p^i (1 - p)^{n-i}$

Random Variables

Random Variables: $X : \Omega \to R$.

Distribution: $Pr[X = a] = \sum_{\omega : X(\omega) = a} Pr(\omega)$

X and Y independent $\iff$ all associated events are independent.
Expectation: $E[X] = \sum_a aPr[X = a] = \sum_{\omega \in \Omega} X(\omega) Pr(\omega)$.
Linearity: $E[X + Y] = E[X] + E[Y]$.

Variance: $Var(X) = E[(X - E[X])^2] = E[X^2] - (E(X))^2$
For independent X, Y, $Var(X + Y) = Var(X) + Var(Y)$.
Also: $Var(cX) = c^2 Var(X)$ and $Var(X + b) = Var(X)$.

Poisson: $X \sim P(\lambda)$ $\quad Pr[X = i] = e^{-\lambda} \frac{\lambda^i}{i!}$.
$E(X) = \lambda$, $Var(X) = \lambda$.
Binomial: $X \sim B(n, p)$ $\quad Pr[X = i] = \binom{n}{i} p^i (1 - p)^{n-i}$
$E(X) = np$, $Var(X) = np(1 - p)$

Random Variables

Random Variables: $X : \Omega \to R$.

Distribution: $Pr[X = a] = \sum_{\omega : X(\omega)=a} Pr(\omega)$

X and Y independent $\iff$ all associated events are independent.
Expectation: $E[X] = \sum_a aPr[X = a] = \sum_{\omega \in \Omega} X(\omega)Pr(\omega)$.
Linearity: $E[X + Y] = E[X] + E[Y]$.

Variance: $Var(X) = E[(X - E[X])^2] = E[X^2] - (E(X))^2$
For independent X, Y, $Var(X + Y) = Var(X) + Var(Y)$.
Also: $Var(cX) = c^2 Var(X)$ and $Var(X + b) = Var(X)$.

Poisson: $X \sim P(\lambda)$ $\quad Pr[X = i] = e^{-\lambda} \frac{\lambda^i}{i!}$.
$E(X) = \lambda$, $Var(X) = \lambda$.
Binomial: $X \sim B(n, p)$ $\quad Pr[X = i] = \binom{n}{i} p^i (1 - p)^{n-i}$
$E(X) = np$, $Var(X) = np(1 - p)$
Uniform: $X \sim U\{1, \ldots, n\}$

Random Variables

Random Variables: $X : \Omega \to R$.

Distribution: $Pr[X = a] = \sum_{\omega : X(\omega)=a} Pr(\omega)$

X and Y independent $\iff$ all associated events are independent.
Expectation: $E[X] = \sum_a aPr[X = a] = \sum_{\omega \in \Omega} X(\omega)Pr(\omega)$.
Linearity: $E[X+Y] = E[X] + E[Y]$.

Variance: $Var(X) = E[(X - E[X])^2] = E[X^2] - (E(X))^2$
For independent X, Y, $Var(X+Y) = Var(X) + Var(Y)$.
Also: $Var(cX) = c^2 Var(X)$ and $Var(X+b) = Var(X)$.

Poisson: $X \sim P(\lambda)$ $\quad Pr[X = i] = e^{-\lambda}\frac{\lambda^i}{i!}$.
$E(X) = \lambda$, $Var(X) = \lambda$.
Binomial: $X \sim B(n, p)$ $\quad Pr[X = i] = \binom{n}{i} p^i (1-p)^{n-i}$
$E(X) = np$, $Var(X) = np(1-p)$
Uniform: $X \sim U\{1, \ldots, n\}$ $\quad \forall i \in [1, n], Pr[X = i] = \frac{1}{n}$.

Random Variables

Random Variables: $X : \Omega \to R$.

Distribution: $Pr[X = a] = \sum_{\omega : X(\omega)=a} Pr(\omega)$

X and Y independent $\iff$ all associated events are independent.
Expectation: $E[X] = \sum_a aPr[X = a] = \sum_{\omega\in\Omega} X(\omega)Pr(\omega)$.
Linearity: $E[X+Y] = E[X] + E[Y]$.

Variance: $Var(X) = E[(X - E[X])^2] = E[X^2] - (E(X))^2$
For independent X, Y, $Var(X+Y) = Var(X) + Var(Y)$.
Also: $Var(cX) = c^2 Var(X)$ and $Var(X+b) = Var(X)$.

Poisson: $X \sim P(\lambda)$ $\quad Pr[X = i] = e^{-\lambda}\frac{\lambda^i}{i!}$.
$E(X) = \lambda$, $Var(X) = \lambda$.
Binomial: $X \sim B(n,p)$ $\quad Pr[X = i] = \binom{n}{i} p^i (1-p)^{n-i}$
$E(X) = np$, $Var(X) = np(1-p)$
Uniform: $X \sim U\{1,\ldots,n\}$ $\quad \forall i \in [1,n], Pr[X = i] = \frac{1}{n}$.
$E[X] = \frac{n+1}{2}$, $Var(X) = \frac{n^2-1}{12}$.

Random Variables

Random Variables: $X : \Omega \to R$.

Distribution: $Pr[X = a] = \sum_{\omega : X(\omega) = a} Pr(\omega)$

X and Y independent $\iff$ all associated events are independent.

Expectation: $E[X] = \sum_a a Pr[X = a] = \sum_{\omega \in \Omega} X(\omega) Pr(\omega)$.

Linearity: $E[X + Y] = E[X] + E[Y]$.

Variance: $Var(X) = E[(X - E[X])^2] = E[X^2] - (E(X))^2$

For independent X, Y, $Var(X + Y) = Var(X) + Var(Y)$.

Also: $Var(cX) = c^2 Var(X)$ and $Var(X + b) = Var(X)$.

Poisson: $X \sim P(\lambda)$ $\quad Pr[X = i] = e^{-\lambda} \frac{\lambda^i}{i!}$.

$E(X) = \lambda$, $Var(X) = \lambda$.

Binomial: $X \sim B(n, p)$ $\quad Pr[X = i] = \binom{n}{i} p^i (1 - p)^{n-i}$

$E(X) = np$, $Var(X) = np(1 - p)$

Uniform: $X \sim U\{1, \ldots, n\}$ $\quad \forall i \in [1, n], Pr[X = i] = \frac{1}{n}$.

$E[X] = \frac{n+1}{2}$, $Var(X) = \frac{n^2 - 1}{12}$.

Geometric: $X \sim G(p)$

Random Variables

Random Variables: $X : \Omega \to R$.

Distribution: $Pr[X = a] = \sum_{\omega : X(\omega)=a} Pr(\omega)$

X and Y independent $\iff$ all associated events are independent.

Expectation: $E[X] = \sum_a a Pr[X = a] = \sum_{\omega \in \Omega} X(\omega) Pr(\omega)$.

Linearity: $E[X + Y] = E[X] + E[Y]$.

Variance: $Var(X) = E[(X - E[X])^2] = E[X^2] - (E(X))^2$

For independent X, Y, $Var(X + Y) = Var(X) + Var(Y)$.

Also: $Var(cX) = c^2 Var(X)$ and $Var(X + b) = Var(X)$.

Poisson: $X \sim P(\lambda)$ $\quad Pr[X = i] = e^{-\lambda} \frac{\lambda^i}{i!}$.

$E(X) = \lambda$, $Var(X) = \lambda$.

Binomial: $X \sim B(n, p)$ $\quad Pr[X = i] = \binom{n}{i} p^i (1-p)^{n-i}$

$E(X) = np$, $Var(X) = np(1-p)$

Uniform: $X \sim U\{1, \ldots, n\}$ $\quad \forall i \in [1, n], Pr[X = i] = \frac{1}{n}$.

$E[X] = \frac{n+1}{2}$, $Var(X) = \frac{n^2-1}{12}$.

Geometric: $X \sim G(p)$ $\quad Pr[X = i] = (1-p)^{i-1} p$

Random Variables

Random Variables: $X : \Omega \to R$.

Distribution: $Pr[X = a] = \sum_{\omega : X(\omega)=a} Pr(\omega)$

X and Y independent $\iff$ all associated events are independent.

Expectation: $E[X] = \sum_a aPr[X = a] = \sum_{\omega \in \Omega} X(\omega)Pr(\omega)$.

Linearity: $E[X + Y] = E[X] + E[Y]$.

Variance: $Var(X) = E[(X - E[X])^2] = E[X^2] - (E(X))^2$

For independent X, Y, $Var(X + Y) = Var(X) + Var(Y)$.

Also: $Var(cX) = c^2 Var(X)$ and $Var(X + b) = Var(X)$.

Poisson: $X \sim P(\lambda)$ $\quad Pr[X = i] = e^{-\lambda} \frac{\lambda^i}{i!}$.

$E(X) = \lambda$, $Var(X) = \lambda$.

Binomial: $X \sim B(n, p)$ $\quad Pr[X = i] = \binom{n}{i} p^i (1 - p)^{n-i}$

$E(X) = np$, $Var(X) = np(1 - p)$

Uniform: $X \sim U\{1, \ldots, n\}$ $\quad \forall i \in [1, n], Pr[X = i] = \frac{1}{n}$.

$E[X] = \frac{n+1}{2}$, $Var(X) = \frac{n^2-1}{12}$.

Geometric: $X \sim G(p)$ $\quad Pr[X = i] = (1 - p)^{i-1} p$

$E(X) = \frac{1}{p}$, $Var(X) = \frac{1-p}{p^2}$

Random Variables

Random Variables: $X : \Omega \to R$.

Distribution: $Pr[X = a] = \sum_{\omega : X(\omega) = a} Pr(\omega)$

X and Y independent $\iff$ all associated events are independent.
Expectation: $E[X] = \sum_a aPr[X = a] = \sum_{\omega \in \Omega} X(\omega)Pr(\omega)$.
Linearity: $E[X + Y] = E[X] + E[Y]$.

Variance: $Var(X) = E[(X - E[X])^2] = E[X^2] - (E(X))^2$
For independent X, Y, $Var(X + Y) = Var(X) + Var(Y)$.
Also: $Var(cX) = c^2 Var(X)$ and $Var(X + b) = Var(X)$.

Poisson: $X \sim P(\lambda)$ $\quad Pr[X = i] = e^{-\lambda} \frac{\lambda^i}{i!}$.
$E(X) = \lambda$, $Var(X) = \lambda$.
Binomial: $X \sim B(n, p)$ $\quad Pr[X = i] = \binom{n}{i} p^i (1-p)^{n-i}$
$E(X) = np$, $Var(X) = np(1-p)$
Uniform: $X \sim U\{1, \ldots, n\}$ $\quad \forall i \in [1, n], Pr[X = i] = \frac{1}{n}$.
$E[X] = \frac{n+1}{2}$, $Var(X) = \frac{n^2-1}{12}$.
Geometric: $X \sim G(p)$ $\quad Pr[X = i] = (1-p)^{i-1} p$
$E(X) = \frac{1}{p}$, $Var(X) = \frac{1-p}{p^2}$

Note: Probability Mass Function $\equiv$ Distribution.

Properties of Covariance

$$cov(X, Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y].$$

Properties of Covariance

$$cov(X, Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y].$$

Fact

(a) $var[X] = cov(X, X)$

Properties of Covariance

$$cov(X, Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y].$$

Fact
(a) $var[X] = cov(X, X)$
(b) X, Y independent $\Rightarrow cov(X, Y) =$

Properties of Covariance

$$cov(X,Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y].$$

Fact
(a) $var[X] = cov(X,X)$
(b) X, Y independent $\Rightarrow cov(X,Y) = 0$

Properties of Covariance

$$cov(X, Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y].$$

Fact

(a) $var[X] = cov(X, X)$

(b) X, Y independent $\Rightarrow cov(X, Y) = 0$

(c) $cov(a + X, b + Y) = cov(X, Y)$

Properties of Covariance

$$cov(X,Y) = E[(X-E[X])(Y-E[Y])] = E[XY] - E[X]E[Y].$$

Fact

(a) $var[X] = cov(X,X)$

(b) X, Y independent $\Rightarrow cov(X,Y) = 0$

(c) $cov(a+X, b+Y) = cov(X,Y)$

(d) $cov(aX+bY, cU+dV) = ac \cdot cov(X,U) + ad \cdot cov(X,V)$
$+ bc \cdot cov(Y,U) + bd \cdot cov(Y,V).$

Properties of Covariance

$$cov(X,Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y].$$

Fact

(a) $var[X] = cov(X,X)$

(b) X, Y independent $\Rightarrow cov(X,Y) = 0$

(c) $cov(a+X, b+Y) = cov(X,Y)$

(d) $$cov(aX+bY, cU+dV) = ac \cdot cov(X,U) + ad \cdot cov(X,V) + bc \cdot cov(Y,U) + bd \cdot cov(Y,V).$$

Proof:

(a)-(b)-(c) are clear.

Properties of Covariance

$$cov(X,Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y].$$

Fact

(a) $var[X] = cov(X,X)$

(b) X, Y independent $\Rightarrow cov(X,Y) = 0$

(c) $cov(a+X, b+Y) = cov(X,Y)$

(d) $$cov(aX+bY, cU+dV) = ac \cdot cov(X,U) + ad \cdot cov(X,V) + bc \cdot cov(Y,U) + bd \cdot cov(Y,V).$$

Proof:

(a)-(b)-(c) are clear.

(d) In view of (c), one can subtract the means and assume that the RVs are zero-mean.

Properties of Covariance

$$cov(X,Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y].$$

Fact

(a) $var[X] = cov(X,X)$

(b) X, Y independent $\Rightarrow cov(X,Y) = 0$

(c) $cov(a + X, b + Y) = cov(X,Y)$

(d) $cov(aX + bY, cU + dV) = ac \cdot cov(X,U) + ad \cdot cov(X,V)$
$\qquad + bc \cdot cov(Y,U) + bd \cdot cov(Y,V).$

Proof:

(a)-(b)-(c) are clear.

(d) In view of (c), one can subtract the means and assume that the RVs are zero-mean. Then,

$$cov(aX + bY, cU + dV) = E[(aX + bY)(cU + dV)]$$

Properties of Covariance

$$cov(X,Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y].$$

Fact

(a) $var[X] = cov(X,X)$

(b) X, Y independent $\Rightarrow cov(X,Y) = 0$

(c) $cov(a+X, b+Y) = cov(X,Y)$

(d) $cov(aX+bY, cU+dV) = ac \cdot cov(X,U) + ad \cdot cov(X,V)$
$\qquad + bc \cdot cov(Y,U) + bd \cdot cov(Y,V).$

Proof:

(a)-(b)-(c) are clear.

(d) In view of (c), one can subtract the means and assume that the RVs are zero-mean. Then,

$$\begin{aligned} cov(aX+bY, cU+dV) &= E[(aX+bY)(cU+dV)] \\ &= ac \cdot E[XU] + ad \cdot E[XV] + bc \cdot E[YU] + bd \cdot E[YV] \end{aligned}$$

Properties of Covariance

$$cov(X,Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y].$$

Fact
(a) $var[X] = cov(X,X)$
(b) X, Y independent $\Rightarrow cov(X,Y) = 0$
(c) $cov(a+X, b+Y) = cov(X,Y)$
(d) $cov(aX+bY, cU+dV) = ac \cdot cov(X,U) + ad \cdot cov(X,V)$
$\qquad + bc \cdot cov(Y,U) + bd \cdot cov(Y,V).$

Proof:
(a)-(b)-(c) are clear.
(d) In view of (c), one can subtract the means and assume that the RVs are zero-mean. Then,

$$\begin{aligned} cov(aX+bY, cU+dV) &= E[(aX+bY)(cU+dV)] \\ &= ac \cdot E[XU] + ad \cdot E[XV] + bc \cdot E[YU] + bd \cdot E[YV] \\ &= ac \cdot cov(X,U) + ad \cdot cov(X,V) + bc \cdot cov(Y,U) + bd \cdot cov(Y,V). \end{aligned}$$

Properties of Covariance

$$cov(X,Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y].$$

Fact

(a) $var[X] = cov(X,X)$

(b) X, Y independent $\Rightarrow cov(X,Y) = 0$

(c) $cov(a + X, b + Y) = cov(X,Y)$

(d) $cov(aX + bY, cU + dV) = ac \cdot cov(X,U) + ad \cdot cov(X,V) + bc \cdot cov(Y,U) + bd \cdot cov(Y,V).$

Proof:

(a)-(b)-(c) are clear.

(d) In view of (c), one can subtract the means and assume that the RVs are zero-mean. Then,

$$\begin{aligned} cov(aX + bY, cU + dV) &= E[(aX + bY)(cU + dV)] \\ &= ac \cdot E[XU] + ad \cdot E[XV] + bc \cdot E[YU] + bd \cdot E[YV] \\ &= ac \cdot cov(X,U) + ad \cdot cov(X,V) + bc \cdot cov(Y,U) + bd \cdot cov(Y,V). \end{aligned}$$

$\square$

Poll.

$$cov(X, Y) = E[XY] - E[X]E[Y] = E[(X - E[X])(Y - E(Y)].$$

Poll.

$cov(X,Y) = E[XY] - E[X]E[Y] = E[(X - E[X])(Y - E(Y)]$.

What is true?

Poll.

$$cov(X,Y) = E[XY] - E[X]E[Y] = E[(X - E[X])(Y - E(Y)].$$

What is true?

(A) $cov(X,X) = Var(X)$.
(B) $cov(X, Y+Z) = cov(XY) + cov(XZ)$
(C) $cov(X,2) = 0$
(D) $cov(X,aX) = a^2 var(X)$

Correlation

Definition The covariance of X and Y is

$$cov(X,Y) := E[(X - E[X])(Y - E[Y])].$$

Correlation

Definition The covariance of X and Y is

$$cov(X, Y) := E[(X - E[X])(Y - E[Y])].$$

Definition The correlation of X, Y, $Cor(X, Y)$ is

$$corr(X, Y) : \frac{cov(X, Y)}{\sigma(X)\sigma(Y)}.$$

Correlation

Definition The covariance of X and Y is

$$cov(X, Y) := E[(X - E[X])(Y - E[Y])].$$

Definition The correlation of X, Y, $Cor(X, Y)$ is

$$corr(X, Y) : \frac{cov(X, Y)}{\sigma(X)\sigma(Y)}.$$

Note: $|corr(X, Y)| \leq 1$.

Correlation

Definition The covariance of X and Y is

$$cov(X, Y) := E[(X - E[X])(Y - E[Y])].$$

Definition The correlation of X, Y, $Cor(X, Y)$ is

$$corr(X, Y) : \frac{cov(X, Y)}{\sigma(X)\sigma(Y)}.$$

Note: $|corr(X, Y)| \leq 1$.

$corr(X, X)$?

Correlation

Definition The covariance of X and Y is

$$cov(X,Y) := E[(X-E[X])(Y-E[Y])].$$

Definition The correlation of X, Y, $Cor(X,Y)$ is

$$corr(X,Y) : \frac{cov(X,Y)}{\sigma(X)\sigma(Y)}.$$

Note: $|corr(X,Y)| \leq 1$.

$corr(X,X)$? 1
$corr(X,-X)$?

Correlation

Definition The covariance of X and Y is

$$cov(X, Y) := E[(X - E[X])(Y - E[Y])].$$

Definition The correlation of X, Y, $Cor(X, Y)$ is

$$corr(X, Y) : \frac{cov(X, Y)}{\sigma(X)\sigma(Y)}.$$

Note: $|corr(X, Y)| \leq 1$.

$corr(X, X)$? 1
$corr(X, -X)$? -1
$corr(X, X/2)$?

Correlation

Definition The covariance of X and Y is

$$cov(X, Y) := E[(X - E[X])(Y - E[Y])].$$

Definition The correlation of X, Y, $Cor(X, Y)$ is

$$corr(X, Y) : \frac{cov(X, Y)}{\sigma(X)\sigma(Y)}.$$

Note: $|corr(X, Y)| \leq 1$.

$corr(X,X)$? 1
$corr(X,-X)$? -1
$corr(X,X/2)$? 1
$corr(X,5X)$?

Correlation

Definition The covariance of X and Y is

$$cov(X,Y) := E[(X - E[X])(Y - E[Y])].$$

Definition The correlation of X, Y, $Cor(X, Y)$ is

$$corr(X,Y) : \frac{cov(X,Y)}{\sigma(X)\sigma(Y)}.$$

Note: $|corr(X,Y)| \leq 1$.

$corr(X,X)$? 1
$corr(X,-X)$? -1
$corr(X,X/2)$? 1
$corr(X,5X)$? 1
$corr(X,X+Y)$ with $var(X) = Var(Y)$, and X, Y independent?

Correlation

Definition The covariance of X and Y is

$$cov(X, Y) := E[(X - E[X])(Y - E[Y])].$$

Definition The correlation of X, Y, $Cor(X, Y)$ is

$$corr(X, Y) : \frac{cov(X, Y)}{\sigma(X)\sigma(Y)}.$$

Note: $|corr(X, Y)| \leq 1$.

$corr(X, X)$? 1
$corr(X, -X)$? -1
$corr(X, X/2)$? 1
$corr(X, 5X)$? 1
$corr(X, X + Y)$ with $var(X) = Var(Y)$, and X, Y independent? $\frac{1}{2}$

Correlation

Definition The covariance of X and Y is

$$cov(X,Y) := E[(X - E[X])(Y - E[Y])].$$

Definition The correlation of X, Y, $Cor(X,Y)$ is

$$corr(X,Y) : \frac{cov(X,Y)}{\sigma(X)\sigma(Y)}.$$

Note: $|corr(X,Y)| \leq 1$.

$corr(X,X)$? 1
$corr(X,-X)$? -1
$corr(X,X/2)$? 1
$corr(X,5X)$? 1
$corr(X,X+Y)$ with $var(X) = Var(Y)$, and X, Y independent? $\frac{1}{2}$

$cov(X,X+Y) = E[(X - E[X])(X - E[X] + Y - E[Y])] =$
$Var(X) + cov(X,Y) = Var(X)$.

Correlation

Definition The covariance of X and Y is

$$cov(X, Y) := E[(X - E[X])(Y - E[Y])].$$

Definition The correlation of X, Y, $Cor(X, Y)$ is

$$corr(X, Y) : \frac{cov(X, Y)}{\sigma(X)\sigma(Y)}.$$

Note: $|corr(X, Y)| \leq 1$.

$corr(X, X)$? 1
$corr(X, -X)$? -1
$corr(X, X/2)$? 1
$corr(X, 5X)$? 1
$corr(X, X+Y)$ with $var(X) = Var(Y)$, and X, Y independent? $\frac{1}{2}$

$cov(X, X+Y) = E[(X - E[X])(X - E[X] + Y - E[Y])] =$
$Var(X) + cov(X, Y) = Var(X)$.

$corr(X, X+Y) = \frac{varX}{\sigma(X)\sigma(X+Y)} = \frac{varX}{\sigma(X)\sqrt{2}\sigma(X)} = \frac{1}{\sqrt{2}}$

Correlation

Definition The covariance of X and Y is

$$cov(X,Y) := E[(X - E[X])(Y - E[Y])].$$

Definition The correlation of X, Y, $Cor(X,Y)$ is

$$corr(X,Y) : \frac{cov(X,Y)}{\sigma(X)\sigma(Y)}.$$

Note: $|corr(X,Y)| \leq 1$.

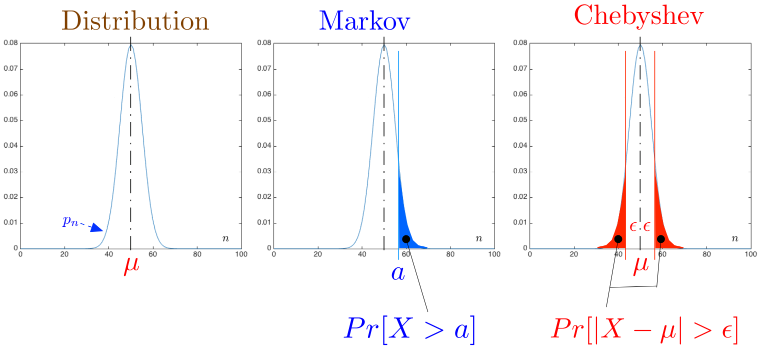
$corr(X,X)$? 1
$corr(X,-X)$? -1
$corr(X,X/2)$? 1
$corr(X,5X)$? 1
$corr(X,X+Y)$ with $var(X) = Var(Y)$, and X, Y independent? $\frac{1}{2}$

$cov(X,X+Y) = E[(X - E[X])(X - E[X] + Y - E[Y])] =$
$Var(X) + cov(X,Y) = Var(X)$.

$corr(X,X+Y) = \frac{varX}{\sigma(X)\sigma(X+Y)} = \frac{varX}{\sigma(X)\sqrt{2}\sigma(X)} = \frac{1}{\sqrt{2}}$

$r^2 = corr(X,Y)^2$ is fraction of variance of Y explained by X.

Inequalities: An Overview



Andrey Markov

Andrey (Andrei) Andreyevich Markov

Born	14 June 1856 N.S. Ryazan, Russian Empire
Died	20 July 1922 (aged 66) Petrograd, Russian SFSR

Andrey Markov

Andrey (Andrei) Andreyevich Markov

Born	14 June 1856 N.S. Ryazan, Russian Empire
Died	20 July 1922 (aged 66) Petrograd, Russian SFSR

Andrey Markov is best known for his work on stochastic processes. A primary subject of his research later became known as Markov chains and Markov processes.

Andrey Markov

Andrey (Andrei) Andreyevich Markov

Born	14 June 1856 N.S. Ryazan, Russian Empire
Died	20 July 1922 (aged 66) Petrograd, Russian SFSR

Andrey Markov is best known for his work on stochastic processes. A primary subject of his research later became known as Markov chains and Markov processes.

Pafnuty Chebyshev was one of his teachers.

Andrey Markov

Andrey (Andrei) Andreyevich Markov

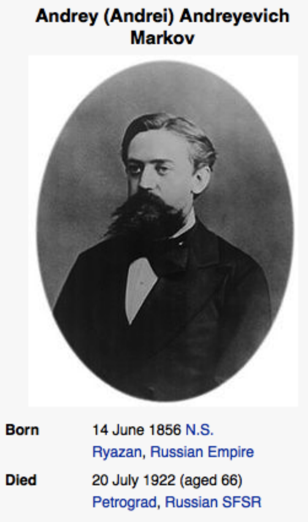
Born	14 June 1856 N.S. Ryazan, Russian Empire
Died	20 July 1922 (aged 66) Petrograd, Russian SFSR

Andrey Markov is best known for his work on stochastic processes. A primary subject of his research later became known as Markov chains and Markov processes.

Pafnuty Chebyshev was one of his teachers.

Markov was an atheist. In 1912 he protested Leo Tolstoy's excommunication from the Russian Orthodox Church by requesting his own excommunication.

Andrey Markov



Andrey Markov is best known for his work on stochastic processes. A primary subject of his research later became known as Markov chains and Markov processes.

Pafnuty Chebyshev was one of his teachers.

Markov was an atheist. In 1912 he protested Leo Tolstoy's excommunication from the Russian Orthodox Church by requesting his own excommunication. The Church complied with his request.

Lake Woebegone: Poll

What is true?

Lake Woebegone: Poll

What is true?

(A) Everyone is above average (on midterm)
(B) For a random variable, at most half the people can be more than twice the average.
(C) For the midterm with no negative scores, at most half the people can be more than twice the average.

Lake Woebegone: Poll

What is true?

(A) Everyone is above average (on midterm)
(B) For a random variable, at most half the people can be more than twice the average.
(C) For the midterm with no negative scores, at most half the people can be more than twice the average.

(A) is false. Average would be higher.
(B) is false. Consder $Pr[X=-2]=1/3$ and $Pr[X=1]=2/3$.
$E[X]=0$.
(C) Is true. Otherwise average would be higher.

Markov's inequality

Markov's inequality

The inequality is named for Andrey Markov, though in work by Pafnuty Chebyshev.

Markov's inequality

The inequality is named for Andrey Markov, though in work by Pafnuty Chebyshev. (Sometimes) called Chebyshev's first inequality.

Markov's inequality

The inequality is named for Andrey Markov, though in work by Pafnuty Chebyshev. (Sometimes) called Chebyshev's first inequality.

Theorem Markov's Inequality

Markov's inequality

The inequality is named for Andrey Markov, though in work by Pafnuty Chebyshev. (Sometimes) called Chebyshev's first inequality.

Theorem Markov's Inequality
Assume $f : \Re \to [0, \infty)$ is nondecreasing.

Markov's inequality

The inequality is named for Andrey Markov, though in work by Pafnuty Chebyshev. (Sometimes) called Chebyshev's first inequality.

Theorem Markov's Inequality
Assume $f : \Re \rightarrow [0, \infty)$ is nondecreasing. Then,

$$Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}, \text{ for all } a \text{ such that } f(a) > 0.$$

Markov's inequality

The inequality is named for Andrey Markov, though in work by Pafnuty Chebyshev. (Sometimes) called Chebyshev's first inequality.

Theorem Markov's Inequality
Assume $f : \Re \rightarrow [0, \infty)$ is nondecreasing. Then,

$$Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}, \text{ for all } a \text{ such that } f(a) > 0.$$

Proof:

Markov's inequality

The inequality is named for Andrey Markov, though in work by Pafnuty Chebyshev. (Sometimes) called Chebyshev's first inequality.

Theorem Markov's Inequality
Assume $f : \Re \to [0, \infty)$ is nondecreasing. Then,

$$Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}, \text{ for all } a \text{ such that } f(a) > 0.$$

Proof:

Claim:

$$1\{X \geq a\} \leq \frac{f(X)}{f(a)}.$$

Markov's inequality

The inequality is named for Andrey Markov, though in work by Pafnuty Chebyshev. (Sometimes) called Chebyshev's first inequality.

Theorem Markov's Inequality
Assume $f : \Re \to [0, \infty)$ is nondecreasing. Then,

$$Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}, \text{ for all } a \text{ such that } f(a) > 0.$$

Proof:

Claim:

$$1\{X \geq a\} \leq \frac{f(X)}{f(a)}.$$

Markov's inequality

The inequality is named for Andrey Markov, though in work by Pafnuty Chebyshev. (Sometimes) called Chebyshev's first inequality.

Theorem Markov's Inequality
Assume $f : \Re \to [0, \infty)$ is nondecreasing. Then,

$$Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}, \text{ for all } a \text{ such that } f(a) > 0.$$

Proof:

Claim:

$$1\{X \geq a\} \leq \frac{f(X)}{f(a)}.$$

If $X < a$, the inequality reads $0 \leq f(x)/f(a)$, since $f(\cdot) \geq 0$.

Markov's inequality

The inequality is named for Andrey Markov, though in work by Pafnuty Chebyshev. (Sometimes) called Chebyshev's first inequality.

Theorem Markov's Inequality
Assume $f : \Re \to [0, \infty)$ is nondecreasing. Then,

$$Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}, \text{ for all } a \text{ such that } f(a) > 0.$$

Proof:

Claim:

$$1\{X \geq a\} \leq \frac{f(X)}{f(a)}.$$

If $X < a$, the inequality reads $0 \leq f(x)/f(a)$, since $f(\cdot) \geq 0$.
If $X \geq a$, it reads $1 \leq f(x)/f(a)$, since $f(\cdot)$ is nondecreasing.

Markov's inequality

The inequality is named for Andrey Markov, though in work by Pafnuty Chebyshev. (Sometimes) called Chebyshev's first inequality.

Theorem Markov's Inequality
Assume $f : \Re \to [0, \infty)$ is nondecreasing. Then,

$$Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}, \text{ for all } a \text{ such that } f(a) > 0.$$

Proof:

Claim:

$$1\{X \geq a\} \leq \frac{f(X)}{f(a)}.$$

If $X < a$, the inequality reads $0 \leq f(x)/f(a)$, since $f(\cdot) \geq 0$.
If $X \geq a$, it reads $1 \leq f(x)/f(a)$, since $f(\cdot)$ is nondecreasing.

Markov's inequality

The inequality is named for Andrey Markov, though in work by Pafnuty Chebyshev. (Sometimes) called Chebyshev's first inequality.

Theorem Markov's Inequality
Assume $f : \Re \to [0, \infty)$ is nondecreasing. Then,

$$Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}, \text{ for all } a \text{ such that } f(a) > 0.$$

Proof:

Claim:

$$1\{X \geq a\} \leq \frac{f(X)}{f(a)}.$$

If $X < a$, the inequality reads $0 \leq f(x)/f(a)$, since $f(\cdot) \geq 0$.
If $X \geq a$, it reads $1 \leq f(x)/f(a)$, since $f(\cdot)$ is nondecreasing.

Taking the expectation yields the inequality,

Markov's inequality

The inequality is named for Andrey Markov, though in work by Pafnuty Chebyshev. (Sometimes) called Chebyshev's first inequality.

Theorem Markov's Inequality
Assume $f : \Re \to [0, \infty)$ is nondecreasing. Then,

$$Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}, \text{ for all } a \text{ such that } f(a) > 0.$$

Proof:

Claim:

$$1\{X \geq a\} \leq \frac{f(X)}{f(a)}.$$

If $X < a$, the inequality reads $0 \leq f(x)/f(a)$, since $f(\cdot) \geq 0$.
If $X \geq a$, it reads $1 \leq f(x)/f(a)$, since $f(\cdot)$ is nondecreasing.

Taking the expectation yields the inequality,
expectation of an indicator is the probability.

Markov's inequality

The inequality is named for Andrey Markov, though in work by Pafnuty Chebyshev. (Sometimes) called Chebyshev's first inequality.

Theorem Markov's Inequality
Assume $f : \Re \to [0, \infty)$ is nondecreasing. Then,

$$Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}, \text{ for all } a \text{ such that } f(a) > 0.$$

Proof:

Claim:

$$1\{X \geq a\} \leq \frac{f(X)}{f(a)}.$$

If $X < a$, the inequality reads $0 \leq f(x)/f(a)$, since $f(\cdot) \geq 0$.
If $X \geq a$, it reads $1 \leq f(x)/f(a)$, since $f(\cdot)$ is nondecreasing.

Taking the expectation yields the inequality,
expectation of an indicator is the probability.
and expectation is monotone, e.g., weighted sum of points.

Markov's inequality

The inequality is named for Andrey Markov, though in work by Pafnuty Chebyshev. (Sometimes) called Chebyshev's first inequality.

Theorem Markov's Inequality
Assume $f : \Re \to [0, \infty)$ is nondecreasing. Then,

$$Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}, \text{ for all } a \text{ such that } f(a) > 0.$$

Proof:

Claim:

$$1\{X \geq a\} \leq \frac{f(X)}{f(a)}.$$

If $X < a$, the inequality reads $0 \leq f(x)/f(a)$, since $f(\cdot) \geq 0$.
If $X \geq a$, it reads $1 \leq f(x)/f(a)$, since $f(\cdot)$ is nondecreasing.

Taking the expectation yields the inequality,
  expectation of an indicator is the probability.
  and expectation is monotone, e.g., weighted sum of points.

Markov's inequality

The inequality is named for Andrey Markov, though in work by Pafnuty Chebyshev. (Sometimes) called Chebyshev's first inequality.

Theorem Markov's Inequality
Assume $f : \Re \to [0, \infty)$ is nondecreasing. Then,

$$Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}, \text{ for all } a \text{ such that } f(a) > 0.$$

Proof:

Claim:

$$1\{X \geq a\} \leq \frac{f(X)}{f(a)}.$$

If $X < a$, the inequality reads $0 \leq f(x)/f(a)$, since $f(\cdot) \geq 0$.
If $X \geq a$, it reads $1 \leq f(x)/f(a)$, since $f(\cdot)$ is nondecreasing.

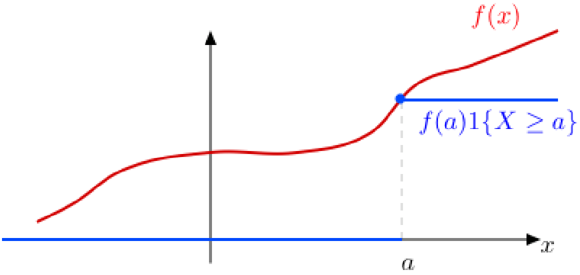
Taking the expectation yields the inequality,
expectation of an indicator is the probability.
and expectation is monotone, e.g., weighted sum of points.

That is, $\sum_v Pr[X = v] 1\{v \geq a\} \leq \sum_v Pr[X = v] \frac{f(v)}{f(a)}$.

$\square$

Intuition: $E[f(X)] \geq f(a) Pr[X > a] = f(a) Pr[X > f(a)]$.

A picture



$$f(a)1\{X \geq a\} \leq f(x) \Rightarrow 1\{X \geq a\} \leq \frac{f(X)}{f(a)}$$

$$\Rightarrow Pr[X \geq a] \leq \frac{E[f(X)]}{f(a)}$$

Markov Inequality Example: G(p)

Let $X = G(p)$.

Markov Inequality Example: G(p)

Let $X = G(p)$. Recall that $E[X] =$

Markov Inequality Example: G(p)

Let $X = G(p)$. Recall that $E[X] = \frac{1}{p}$ and $E[X^2] =$

Markov Inequality Example: G(p)

Let $X = G(p)$. Recall that $E[X] = \frac{1}{p}$ and $E[X^2] = \frac{2-p}{p^2}$.

Markov Inequality Example: G(p)

Let $X = G(p)$. Recall that $E[X] = \frac{1}{p}$ and $E[X^2] = \frac{2-p}{p^2}$.

Choosing $f(x) = x$, we get

Markov Inequality Example: G(p)

Let $X = G(p)$. Recall that $E[X] = \frac{1}{p}$ and $E[X^2] = \frac{2-p}{p^2}$.

Choosing $f(x) = x$, we get

$$Pr[X \geq a] \leq \frac{E[X]}{a} = \frac{1}{ap}.$$

Markov Inequality Example: G(p)

Let $X = G(p)$. Recall that $E[X] = \frac{1}{p}$ and $E[X^2] = \frac{2-p}{p^2}$.

Choosing $f(x) = x$, we get

$$Pr[X \geq a] \leq \frac{E[X]}{a} = \frac{1}{ap}.$$

Choosing $f(x) = x^2$, we get

Markov Inequality Example: G(p)

Let $X = G(p)$. Recall that $E[X] = \frac{1}{p}$ and $E[X^2] = \frac{2-p}{p^2}$.

Choosing $f(x) = x$, we get

$$Pr[X \geq a] \leq \frac{E[X]}{a} = \frac{1}{ap}.$$

Choosing $f(x) = x^2$, we get

$$Pr[X \geq a] \leq \frac{E[X^2]}{a^2} = \frac{2-p}{p^2 a^2}.$$

Markov Inequality Example: G(p)

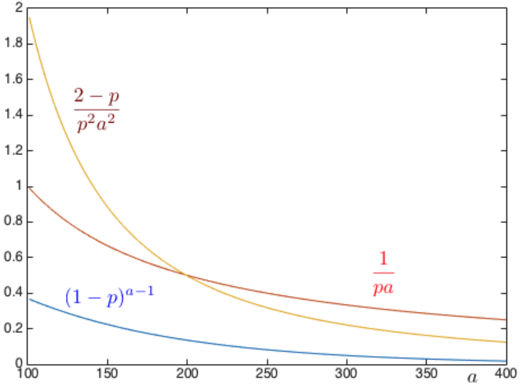
Let $X = G(p)$. Recall that $E[X] = \frac{1}{p}$ and $E[X^2] = \frac{2-p}{p^2}$.

Choosing $f(x) = x$, we get

$$Pr[X \geq a] \leq \frac{E[X]}{a} = \frac{1}{ap}.$$

Choosing $f(x) = x^2$, we get

$$Pr[X \geq a] \leq \frac{E[X^2]}{a^2} = \frac{2-p}{p^2 a^2}.$$



Markov Inequality Example: $P(\lambda)$

Let $X = P(\lambda)$.

Markov Inequality Example: $P(\lambda)$

Let $X = P(\lambda)$. Recall that $E[X] =$

Markov Inequality Example: $P(\lambda)$

Let $X = P(\lambda)$. Recall that $E[X] = \lambda$ and $E[X^2] =$

Markov Inequality Example: $P(\lambda)$

Let $X = P(\lambda)$. Recall that $E[X] = \lambda$ and $E[X^2] = \lambda + \lambda^2$.

Markov Inequality Example: $P(\lambda)$

Let $X = P(\lambda)$. Recall that $E[X] = \lambda$ and $E[X^2] = \lambda + \lambda^2$.

Choosing $f(x) = x$, we
get

Markov Inequality Example: $P(\lambda)$

Let $X = P(\lambda)$. Recall that $E[X] = \lambda$ and $E[X^2] = \lambda + \lambda^2$.

Choosing $f(x) = x$, we get

$$Pr[X \geq a] \leq \frac{E[X]}{a} = \frac{\lambda}{a}.$$

Markov Inequality Example: $P(\lambda)$

Let $X = P(\lambda)$. Recall that $E[X] = \lambda$ and $E[X^2] = \lambda + \lambda^2$.

Choosing $f(x) = x$, we get

$$Pr[X \geq a] \leq \frac{E[X]}{a} = \frac{\lambda}{a}.$$

Choosing $f(x) = x^2$, we get

Markov Inequality Example: $P(\lambda)$

Let $X = P(\lambda)$. Recall that $E[X] = \lambda$ and $E[X^2] = \lambda + \lambda^2$.

Choosing $f(x) = x$, we get

$$Pr[X \geq a] \leq \frac{E[X]}{a} = \frac{\lambda}{a}.$$

Choosing $f(x) = x^2$, we get

$$Pr[X \geq a] \leq \frac{E[X^2]}{a^2} = \frac{\lambda + \lambda^2}{a^2}.$$

Markov Inequality Example: $P(\lambda)$

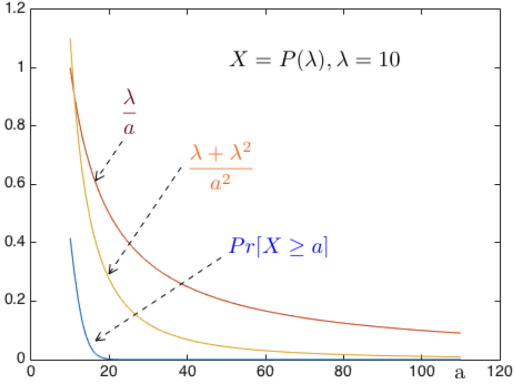
Let $X = P(\lambda)$. Recall that $E[X] = \lambda$ and $E[X^2] = \lambda + \lambda^2$.

Choosing $f(x) = x$, we get

$$Pr[X \geq a] \leq \frac{E[X]}{a} = \frac{\lambda}{a}.$$

Choosing $f(x) = x^2$, we get

$$Pr[X \geq a] \leq \frac{E[X^2]}{a^2} = \frac{\lambda + \lambda^2}{a^2}.$$



Chebyshev's Inequality

This is Pafnuty's inequality:

Chebyshev's Inequality

This is Pafnuty's inequality:
Theorem:

$$Pr[|X - E[X]| > a] \leq \frac{var[X]}{a^2}, \text{ for all } a > 0.$$

Chebyshev's Inequality

This is Pafnuty's inequality:
Theorem:

$$Pr[|X - E[X]| > a] \le \frac{var[X]}{a^2}, \text{ for all } a > 0.$$

Proof: Let $Y = |X - E[X]|$ and $f(y) = y^2$.

Chebyshev's Inequality

This is Pafnuty's inequality:
Theorem:

$$Pr[|X - E[X]| > a] \leq \frac{var[X]}{a^2}, \text{ for all } a > 0.$$

Proof: Let $Y = |X - E[X]|$ and $f(y) = y^2$. Then,

$$Pr[Y \geq a] \leq \frac{E[f(Y)]}{f(a)}$$

Chebyshev's Inequality

This is Pafnuty's inequality:
Theorem:

$$Pr[|X - E[X]| > a] \leq \frac{var[X]}{a^2}, \text{ for all } a > 0.$$

Proof: Let $Y = |X - E[X]|$ and $f(y) = y^2$. Then,

$$Pr[Y \geq a] \leq \frac{E[f(Y)]}{f(a)} = \frac{var[X]}{a^2}.$$

$\square$

Chebyshev's Inequality

This is Pafnuty's inequality:
Theorem:

$$Pr[|X - E[X]| > a] \leq \frac{var[X]}{a^2}, \text{ for all } a > 0.$$

Proof: Let $Y = |X - E[X]|$ and $f(y) = y^2$. Then,

$$Pr[Y \geq a] \leq \frac{E[f(Y)]}{f(a)} = \frac{var[X]}{a^2}.$$

$\square$

This result confirms that the variance measures the "deviations from the mean."

Chebyshev and Poisson

Chebyshev and Poisson

Let $X = P(\lambda)$. Then, $E[X] = \lambda$ and $var[X] =$

Chebyshev and Poisson

Let $X = P(\lambda)$. Then, $E[X] = \lambda$ and $var[X] = \lambda$.

Chebyshev and Poisson

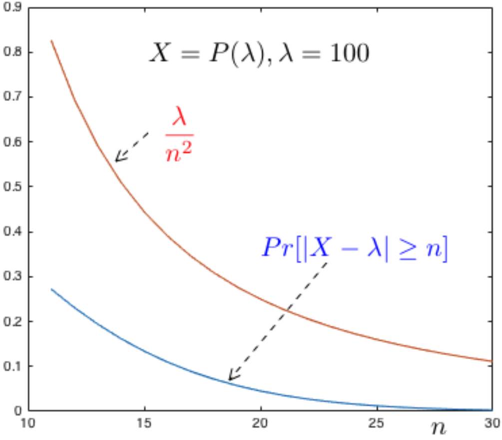
Let $X = P(\lambda)$. Then, $E[X] = \lambda$ and $var[X] = \lambda$. Thus,

$$Pr[|X - \lambda| \geq n] \leq \frac{var[X]}{n^2} = \frac{\lambda}{n^2}.$$

Chebyshev and Poisson

Let $X = P(\lambda)$. Then, $E[X] = \lambda$ and $var[X] = \lambda$. Thus,

$$Pr[|X - \lambda| \geq n] \leq \frac{var[X]}{n^2} = \frac{\lambda}{n^2}.$$



Chebyshev and Poisson (continued)

Let $X = P(\lambda)$. Then, $E[X] = \lambda$ and $var[X] = \lambda$.

Chebyshev and Poisson (continued)

Let $X = P(\lambda)$. Then, $E[X] = \lambda$ and $var[X] = \lambda$. By Markov's inequality,

$$Pr[X \geq a] \leq \frac{E[X^2]}{a^2} = \frac{\lambda + \lambda^2}{a^2}.$$

Chebyshev and Poisson (continued)

Let $X = P(\lambda)$. Then, $E[X] = \lambda$ and $var[X] = \lambda$. By Markov's inequality,

$$Pr[X \geq a] \leq \frac{E[X^2]}{a^2} = \frac{\lambda + \lambda^2}{a^2}.$$

Also, if $a > \lambda$, then $X \geq a \Rightarrow X - \lambda \geq a - \lambda > 0$

Chebyshev and Poisson (continued)

Let $X = P(\lambda)$. Then, $E[X] = \lambda$ and $var[X] = \lambda$. By Markov's inequality,

$$Pr[X \geq a] \leq \frac{E[X^2]}{a^2} = \frac{\lambda + \lambda^2}{a^2}.$$

Also, if $a > \lambda$, then $X \geq a \Rightarrow X - \lambda \geq a - \lambda > 0 \Rightarrow |X - \lambda| \geq a - \lambda$.

Chebyshev and Poisson (continued)

Let $X = P(\lambda)$. Then, $E[X] = \lambda$ and $var[X] = \lambda$. By Markov's inequality,

$$Pr[X \geq a] \leq \frac{E[X^2]}{a^2} = \frac{\lambda + \lambda^2}{a^2}.$$

Also, if $a > \lambda$, then $X \geq a \Rightarrow X - \lambda \geq a - \lambda > 0 \Rightarrow |X - \lambda| \geq a - \lambda$.

Hence, for $a > \lambda$,

Chebyshev and Poisson (continued)

Let $X = P(\lambda)$. Then, $E[X] = \lambda$ and $var[X] = \lambda$. By Markov's inequality,

$$Pr[X \geq a] \leq \frac{E[X^2]}{a^2} = \frac{\lambda + \lambda^2}{a^2}.$$

Also, if $a > \lambda$, then $X \geq a \Rightarrow X - \lambda \geq a - \lambda > 0 \Rightarrow |X - \lambda| \geq a - \lambda$.

Hence, for $a > \lambda$, $Pr[X \geq a] \leq Pr[|X - \lambda| \geq a - \lambda] \leq \frac{\lambda}{(a-\lambda)^2}$.

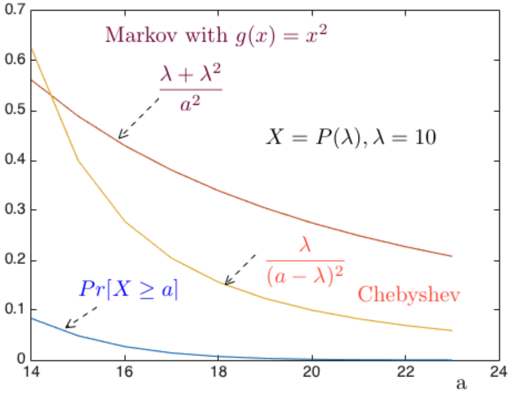
Chebyshev and Poisson (continued)

Let $X = P(\lambda)$. Then, $E[X] = \lambda$ and $var[X] = \lambda$. By Markov's inequality,

$$Pr[X \geq a] \leq \frac{E[X^2]}{a^2} = \frac{\lambda + \lambda^2}{a^2}.$$

Also, if $a > \lambda$, then $X \geq a \Rightarrow X - \lambda \geq a - \lambda > 0 \Rightarrow |X - \lambda| \geq a - \lambda$.

Hence, for $a > \lambda$, $Pr[X \geq a] \leq Pr[|X - \lambda| \geq a - \lambda] \leq \frac{\lambda}{(a-\lambda)^2}$.



Fraction of H's

Here is a classical application of Chebyshev's inequality.

Fraction of H's

Here is a classical application of Chebyshev's inequality.

How likely is it that the fraction of H's differs from 50%?

Fraction of H's

Here is a classical application of Chebyshev's inequality.

How likely is it that the fraction of H's differs from 50%?

Let $X_m = 1$ if the m-th flip of a fair coin is H and $X_m = 0$ otherwise.

Fraction of H's

Here is a classical application of Chebyshev's inequality.

How likely is it that the fraction of H's differs from 50%?

Let $X_m = 1$ if the m-th flip of a fair coin is H and $X_m = 0$ otherwise.

Define

$$Y_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

Fraction of H's

Here is a classical application of Chebyshev's inequality.

How likely is it that the fraction of H's differs from 50%?

Let $X_m = 1$ if the m-th flip of a fair coin is H and $X_m = 0$ otherwise.

Define

$$Y_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

We want to estimate

$$Pr[|Y_n - 0.5| \geq 0.1] = Pr[Y_n \leq 0.4 \text{ or } Y_n \geq 0.6].$$

Fraction of H's

Here is a classical application of Chebyshev's inequality.

How likely is it that the fraction of H's differs from 50%?

Let $X_m = 1$ if the m-th flip of a fair coin is H and $X_m = 0$ otherwise.

Define

$$Y_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

We want to estimate

$$Pr[|Y_n - 0.5| \geq 0.1] = Pr[Y_n \leq 0.4 \text{ or } Y_n \geq 0.6].$$

By Chebyshev,

Fraction of *H*'s

Here is a classical application of Chebyshev's inequality.

How likely is it that the fraction of *H*'s differs from 50%?

Let $X_m = 1$ if the m-th flip of a fair coin is H and $X_m = 0$ otherwise.

Define

$$Y_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

We want to estimate

$$Pr[|Y_n - 0.5| \geq 0.1] = Pr[Y_n \leq 0.4 \text{ or } Y_n \geq 0.6].$$

By Chebyshev,

$$Pr[|Y_n - 0.5| \geq 0.1] \leq \frac{var[Y_n]}{(0.1)^2} = 100 var[Y_n].$$

Fraction of H's

Here is a classical application of Chebyshev's inequality.

How likely is it that the fraction of H's differs from 50%?

Let $X_m = 1$ if the m-th flip of a fair coin is H and $X_m = 0$ otherwise.

Define

$$Y_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

We want to estimate

$$Pr[|Y_n - 0.5| \geq 0.1] = Pr[Y_n \leq 0.4 \text{ or } Y_n \geq 0.6].$$

By Chebyshev,

$$Pr[|Y_n - 0.5| \geq 0.1] \leq \frac{var[Y_n]}{(0.1)^2} = 100 var[Y_n].$$

Now,

Fraction of H's

Here is a classical application of Chebyshev's inequality.

How likely is it that the fraction of H's differs from 50%?

Let $X_m = 1$ if the m-th flip of a fair coin is H and $X_m = 0$ otherwise.

Define

$$Y_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

We want to estimate

$$Pr[|Y_n - 0.5| \geq 0.1] = Pr[Y_n \leq 0.4 \text{ or } Y_n \geq 0.6].$$

By Chebyshev,

$$Pr[|Y_n - 0.5| \geq 0.1] \leq \frac{var[Y_n]}{(0.1)^2} = 100 var[Y_n].$$

Now,

$$var[Y_n] = \frac{1}{n^2}(var[X_1] + \cdots + var[X_n])$$

Fraction of H's

Here is a classical application of Chebyshev's inequality.

How likely is it that the fraction of H's differs from 50%?

Let $X_m = 1$ if the m-th flip of a fair coin is H and $X_m = 0$ otherwise.

Define

$$Y_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

We want to estimate

$$Pr[|Y_n - 0.5| \geq 0.1] = Pr[Y_n \leq 0.4 \text{ or } Y_n \geq 0.6].$$

By Chebyshev,

$$Pr[|Y_n - 0.5| \geq 0.1] \leq \frac{var[Y_n]}{(0.1)^2} = 100 var[Y_n].$$

Now,

$$var[Y_n] = \frac{1}{n^2}(var[X_1] + \cdots + var[X_n]) = \frac{1}{n} var[X_1]$$

Fraction of H's

Here is a classical application of Chebyshev's inequality.

How likely is it that the fraction of H's differs from 50%?

Let $X_m = 1$ if the m-th flip of a fair coin is H and $X_m = 0$ otherwise.

Define

$$Y_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

We want to estimate

$$Pr[|Y_n - 0.5| \geq 0.1] = Pr[Y_n \leq 0.4 \text{ or } Y_n \geq 0.6].$$

By Chebyshev,

$$Pr[|Y_n - 0.5| \geq 0.1] \leq \frac{var[Y_n]}{(0.1)^2} = 100 var[Y_n].$$

Now,

$$var[Y_n] = \frac{1}{n^2}(var[X_1] + \cdots + var[X_n]) = \frac{1}{n} var[X_1] \leq \frac{1}{4n}.$$

Fraction of H's

Here is a classical application of Chebyshev's inequality.

How likely is it that the fraction of H's differs from 50%?

Let $X_m = 1$ if the m-th flip of a fair coin is H and $X_m = 0$ otherwise.

Define

$$Y_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

We want to estimate

$$Pr[|Y_n - 0.5| \geq 0.1] = Pr[Y_n \leq 0.4 \text{ or } Y_n \geq 0.6].$$

By Chebyshev,

$$Pr[|Y_n - 0.5| \geq 0.1] \leq \frac{var[Y_n]}{(0.1)^2} = 100 var[Y_n].$$

Now,

$$var[Y_n] = \frac{1}{n^2}(var[X_1] + \cdots + var[X_n]) = \frac{1}{n} var[X_1] \leq \frac{1}{4n}.$$

$Var(X_i) = p(1 - lp) \leq (.5)(.5) = \frac{1}{4}$

Fraction of *H*'s

$$Y_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

$$Pr[|Y_n - 0.5| \geq 0.1] \leq \frac{25}{n}.$$

Fraction of H's

$$Y_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

$$Pr[|Y_n - 0.5| \geq 0.1] \leq \frac{25}{n}.$$

For $n = 1,000$, we find that this probability is less than 2.5%.

Fraction of H's

$$Y_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

$$Pr[|Y_n - 0.5| \geq 0.1] \leq \frac{25}{n}.$$

For $n = 1,000$, we find that this probability is less than 2.5%.

As $n \to \infty$, this probability goes to zero.

Fraction of H's

$$Y_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

$$Pr[|Y_n - 0.5| \geq 0.1] \leq \frac{25}{n}.$$

For $n = 1,000$, we find that this probability is less than 2.5%.

As $n \to \infty$, this probability goes to zero.

In fact, for any $\varepsilon > 0$,

Fraction of H's

$$Y_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

$$Pr[|Y_n - 0.5| \geq 0.1] \leq \frac{25}{n}.$$

For $n = 1,000$, we find that this probability is less than 2.5%.

As $n \to \infty$, this probability goes to zero.

In fact, for any $\varepsilon > 0$, as $n \to \infty$,

Fraction of H's

$$Y_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

$$Pr[|Y_n - 0.5| \geq 0.1] \leq \frac{25}{n}.$$

For $n = 1,000$, we find that this probability is less than 2.5%.

As $n \to \infty$, this probability goes to zero.

In fact, for any $\varepsilon > 0$, as $n \to \infty$, the probability that the fraction of Hs is within $\varepsilon > 0$ of 50% approaches 1:

Fraction of H's

$$Y_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

$$Pr[|Y_n - 0.5| \geq 0.1] \leq \frac{25}{n}.$$

For $n = 1,000$, we find that this probability is less than 2.5%.

As $n \to \infty$, this probability goes to zero.

In fact, for any $\varepsilon > 0$, as $n \to \infty$, the probability that the fraction of Hs is within $\varepsilon > 0$ of 50% approaches 1:

$$Pr[|Y_n - 0.5| \leq \varepsilon] \to 1.$$

Fraction of H's

$$Y_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

$$Pr[|Y_n - 0.5| \geq 0.1] \leq \frac{25}{n}.$$

For $n = 1,000$, we find that this probability is less than 2.5%.

As $n \to \infty$, this probability goes to zero.

In fact, for any $\varepsilon > 0$, as $n \to \infty$, the probability that the fraction of Hs is within $\varepsilon > 0$ of 50% approaches 1:

$$Pr[|Y_n - 0.5| \leq \varepsilon] \to 1.$$

This is an example of the Law of Large Numbers.

Fraction of H's

$$Y_n = \frac{X_1 + \cdots + X_n}{n}, \text{ for } n \geq 1.$$

$$Pr[|Y_n - 0.5| \geq 0.1] \leq \frac{25}{n}.$$

For $n = 1,000$, we find that this probability is less than 2.5%.

As $n \to \infty$, this probability goes to zero.

In fact, for any $\varepsilon > 0$, as $n \to \infty$, the probability that the fraction of Hs is within $\varepsilon > 0$ of 50% approaches 1:

$$Pr[|Y_n - 0.5| \leq \varepsilon] \to 1.$$

This is an example of the Law of Large Numbers.

We look at a general case next.

Weak Law of Large Numbers

Theorem Weak Law of Large Numbers

Weak Law of Large Numbers

Theorem Weak Law of Large Numbers

Let $X_1, X_2, \ldots$ be pairwise independent with the same distribution and mean μ.

Weak Law of Large Numbers

Theorem Weak Law of Large Numbers

Let $X_1, X_2, \ldots$ be pairwise independent with the same distribution and mean μ. Then, for all $\varepsilon > 0$,

$$Pr[|\frac{X_1 + \cdots + X_n}{n} - \mu| \geq \varepsilon] \to 0, \text{ as } n \to \infty.$$

Weak Law of Large Numbers

Theorem Weak Law of Large Numbers

Let $X_1, X_2, \ldots$ be pairwise independent with the same distribution and mean μ. Then, for all $\varepsilon > 0$,

$$Pr[|\frac{X_1 + \cdots + X_n}{n} - \mu| \geq \varepsilon] \to 0, \text{ as } n \to \infty.$$

Proof:
Let $Y_n = \frac{X_1 + \cdots + X_n}{n}$. Then

Weak Law of Large Numbers

Theorem Weak Law of Large Numbers

Let $X_1, X_2, \ldots$ be pairwise independent with the same distribution and mean μ. Then, for all $\varepsilon > 0$,

$$Pr[|\frac{X_1 + \cdots + X_n}{n} - \mu| \geq \varepsilon] \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Proof:
Let $Y_n = \frac{X_1 + \cdots + X_n}{n}$. Then

$$Pr[|Y_n - \mu| \geq \varepsilon] \quad \leq \quad \frac{var[Y_n]}{\varepsilon^2}$$

Weak Law of Large Numbers

Theorem Weak Law of Large Numbers

Let $X_1, X_2, \ldots$ be pairwise independent with the same distribution and mean μ. Then, for all $\varepsilon > 0$,

$$Pr[|\frac{X_1 + \cdots + X_n}{n} - \mu| \geq \varepsilon] \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Proof:
Let $Y_n = \frac{X_1 + \cdots + X_n}{n}$. Then

$$Pr[|Y_n - \mu| \geq \varepsilon] \quad \leq \quad \frac{var[Y_n]}{\varepsilon^2} = \frac{var[X_1 + \cdots + X_n]}{n^2 \varepsilon^2}$$

Weak Law of Large Numbers

Theorem Weak Law of Large Numbers

Let $X_1, X_2, \ldots$ be pairwise independent with the same distribution and mean μ. Then, for all $\varepsilon > 0$,

$$Pr[|\frac{X_1 + \cdots + X_n}{n} - \mu| \geq \varepsilon] \to 0, \text{ as } n \to \infty.$$

Proof:
Let $Y_n = \frac{X_1 + \cdots + X_n}{n}$. Then

$$\begin{aligned} Pr[|Y_n - \mu| \geq \varepsilon] &\leq \frac{var[Y_n]}{\varepsilon^2} = \frac{var[X_1 + \cdots + X_n]}{n^2\varepsilon^2} \\ &= \frac{nvar[X_1]}{n^2\varepsilon^2} \end{aligned}$$

Weak Law of Large Numbers

Theorem Weak Law of Large Numbers

Let $X_1, X_2, \ldots$ be pairwise independent with the same distribution and mean μ. Then, for all $\varepsilon > 0$,

$$Pr[|\frac{X_1 + \cdots + X_n}{n} - \mu| \geq \varepsilon] \to 0, \text{ as } n \to \infty.$$

Proof:
Let $Y_n = \frac{X_1 + \cdots + X_n}{n}$. Then

$$\begin{aligned} Pr[|Y_n - \mu| \geq \varepsilon] &\leq \frac{var[Y_n]}{\varepsilon^2} = \frac{var[X_1 + \cdots + X_n]}{n^2 \varepsilon^2} \\ &= \frac{n var[X_1]}{n^2 \varepsilon^2} = \frac{var[X_1]}{n \varepsilon^2} \end{aligned}$$

Weak Law of Large Numbers

Theorem Weak Law of Large Numbers

Let $X_1, X_2, \ldots$ be pairwise independent with the same distribution and mean μ. Then, for all $\varepsilon > 0$,

$$Pr[|\frac{X_1 + \cdots + X_n}{n} - \mu| \geq \varepsilon] \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Proof:
Let $Y_n = \frac{X_1 + \cdots + X_n}{n}$. Then

$$\begin{aligned} Pr[|Y_n - \mu| \geq \varepsilon] &\leq \frac{var[Y_n]}{\varepsilon^2} = \frac{var[X_1 + \cdots + X_n]}{n^2\varepsilon^2} \\ &= \frac{nvar[X_1]}{n^2\varepsilon^2} = \frac{var[X_1]}{n\varepsilon^2} \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

$\square$

Summary

$\boxed{\text{Variance; Inequalities; WLLN}}$

Summary

$\boxed{\text{Variance; Inequalities; WLLN}}$

- Variance: $var[X] := E[(X - E[X])^2] = E[X^2] - E[X]^2$

Summary

$\boxed{\text{Variance; Inequalities; WLLN}}$

- Variance: $var[X] := E[(X - E[X])^2] = E[X^2] - E[X]^2$
- Fact: $var[aX + b]a^2 var[X]$

Summary

$\boxed{\text{Variance; Inequalities; WLLN}}$

- Variance: $var[X] := E[(X - E[X])^2] = E[X^2] - E[X]^2$
- Fact: $var[aX + b]a^2 var[X]$
- Sum: X, Y, Z pairwise ind. $\Rightarrow var[X + Y + Z] = \cdots$

Summary

$\boxed{\text{Variance; Inequalities; WLLN}}$

- Variance: $var[X] := E[(X-E[X])^2] = E[X^2] - E[X]^2$
- Fact: $var[aX+b]a^2 var[X]$
- Sum: X, Y, Z pairwise ind. $\Rightarrow var[X+Y+Z] = \cdots$
- Markov: $Pr[X \geq a] \leq E[f(X)]/f(a)$ where ...

Summary

$\boxed{\text{Variance; Inequalities; WLLN}}$

- Variance: $var[X] := E[(X - E[X])^2] = E[X^2] - E[X]^2$
- Fact: $var[aX + b]a^2 var[X]$
- Sum: X, Y, Z pairwise ind. $\Rightarrow var[X + Y + Z] = \cdots$
- Markov: $Pr[X \geq a] \leq E[f(X)]/f(a)$ where ...
- Chebyshev: $Pr[|X - E[X]| \geq a] \leq var[X]/a^2$

Summary

Variance; Inequalities; WLLN

- Variance: $var[X] := E[(X - E[X])^2] = E[X^2] - E[X]^2$
- Fact: $var[aX + b]a^2 var[X]$
- Sum: X, Y, Z pairwise ind. $\Rightarrow var[X + Y + Z] = \cdots$
- Markov: $Pr[X \geq a] \leq E[f(X)]/f(a)$ where ...
- Chebyshev: $Pr[|X - E[X]| \geq a] \leq var[X]/a^2$
- WLLN: X_m i.i.d. $\Rightarrow \frac{X_1 + \cdots + X_n}{n} \approx E[X]$

Confidence?

Confidence?

- You flip a coin once and get H.

Confidence?

- You flip a coin once and get H.

  Do think that $Pr[H] = 1$?

Confidence?

- You flip a coin once and get H.

  Do think that $Pr[H] = 1$?
- You flip a coin 10 times and get 5 Hs.

Confidence?

- You flip a coin once and get H.

  Do think that $Pr[H] = 1$?
- You flip a coin 10 times and get 5 Hs.

  Are you sure that $Pr[H] = 0.5$?

Confidence?

- You flip a coin once and get H.

  Do think that $Pr[H] = 1$?
- You flip a coin 10 times and get 5 Hs.

  Are you sure that $Pr[H] = 0.5$?
- You flip a coin 10^6 times and get 35% of Hs.

Confidence?

- You flip a coin once and get H.

  Do think that $Pr[H] = 1$?
- You flip a coin 10 times and get 5 Hs.

  Are you sure that $Pr[H] = 0.5$?
- You flip a coin 10^6 times and get 35% of Hs.

  How much are you willing to bet that $Pr[H]$ is exactly 0.35?

Confidence?

- You flip a coin once and get H.

  Do think that $Pr[H] = 1$?
- You flip a coin 10 times and get 5 Hs.

  Are you sure that $Pr[H] = 0.5$?
- You flip a coin 10^6 times and get 35% of Hs.

  How much are you willing to bet that $Pr[H]$ is exactly 0.35?

  How much are you willing to bet that $Pr[H] \in [0.3, 0.4]$?

Confidence?

- You flip a coin once and get H.

  Do think that $Pr[H] = 1$?
- You flip a coin 10 times and get 5 Hs.

  Are you sure that $Pr[H] = 0.5$?
- You flip a coin 10^6 times and get 35% of Hs.

  How much are you willing to bet that $Pr[H]$ is exactly 0.35?

  How much are you willing to bet that $Pr[H] \in [0.3, 0.4]$?

  Did different exam rooms perform differently?

Confidence?

- You flip a coin once and get H.

  Do think that $Pr[H] = 1$?
- You flip a coin 10 times and get 5 Hs.

  Are you sure that $Pr[H] = 0.5$?
- You flip a coin 10^6 times and get 35% of Hs.

  How much are you willing to bet that $Pr[H]$ is exactly 0.35?

  How much are you willing to bet that $Pr[H] \in [0.3, 0.4]$?

  Did different exam rooms perform differently?

More generally, you estimate an unknown quantity θ.

Confidence?

- You flip a coin once and get H.

  Do think that $Pr[H] = 1$?
- You flip a coin 10 times and get 5 Hs.

  Are you sure that $Pr[H] = 0.5$?
- You flip a coin 10^6 times and get 35% of Hs.

  How much are you willing to bet that $Pr[H]$ is exactly 0.35?

  How much are you willing to bet that $Pr[H] \in [0.3, 0.4]$?

  Did different exam rooms perform differently?

More generally, you estimate an unknown quantity θ.

Your estimate is $\hat{\theta}$.

Confidence?

- You flip a coin once and get H.

  Do think that $Pr[H] = 1$?
- You flip a coin 10 times and get 5 Hs.

  Are you sure that $Pr[H] = 0.5$?
- You flip a coin 10^6 times and get 35% of Hs.

  How much are you willing to bet that $Pr[H]$ is exactly 0.35?

  How much are you willing to bet that $Pr[H] \in [0.3, 0.4]$?

  Did different exam rooms perform differently?

More generally, you estimate an unknown quantity θ.

Your estimate is $\hat{\theta}$.

How much confidence do you have in your estimate?

Confidence?

Confidence?

Confidence is essential is many applications:

Confidence?

Confidence is essential is many applications:

- How effective is a medication?

Confidence?

Confidence is essential is many applications:

- How effective is a medication?
- Are we sure of the mileage of a car?

Confidence?

Confidence is essential is many applications:

- How effective is a medication?
- Are we sure of the mileage of a car?
- Can we guarantee the lifespan of a device?

Confidence?

Confidence is essential is many applications:

- How effective is a medication?
- Are we sure of the mileage of a car?
- Can we guarantee the lifespan of a device?
- We simulated a system. Do we trust the simulation results?

Confidence?

Confidence is essential is many applications:

- How effective is a medication?
- Are we sure of the mileage of a car?
- Can we guarantee the lifespan of a device?
- We simulated a system. Do we trust the simulation results?
- Is an algorithm guaranteed to be fast?

Confidence?

Confidence is essential is many applications:

- How effective is a medication?
- Are we sure of the mileage of a car?
- Can we guarantee the lifespan of a device?
- We simulated a system. Do we trust the simulation results?
- Is an algorithm guaranteed to be fast?
- Do we know that a program has no bug?

Confidence?

Confidence is essential is many applications:

- How effective is a medication?
- Are we sure of the mileage of a car?
- Can we guarantee the lifespan of a device?
- We simulated a system. Do we trust the simulation results?
- Is an algorithm guaranteed to be fast?
- Do we know that a program has no bug?

As scientists and engineers and voters, be convinced of this fact:

Confidence?

Confidence is essential is many applications:

- How effective is a medication?
- Are we sure of the mileage of a car?
- Can we guarantee the lifespan of a device?
- We simulated a system. Do we trust the simulation results?
- Is an algorithm guaranteed to be fast?
- Do we know that a program has no bug?

As scientists and engineers and voters, be convinced of this fact:

An estimate without confidence level is useless!

Confidence Interval

Confidence Interval

The following definition captures precisely the notion of confidence.

Confidence Interval

The following definition captures precisely the notion of confidence.

Definition: Confidence Interval

Confidence Interval

The following definition captures precisely the notion of confidence.

Definition: Confidence Interval

An interval $[a, b]$ is a 95%-confidence interval for an unknown quantity θ if

Confidence Interval

The following definition captures precisely the notion of confidence.

Definition: Confidence Interval

An interval $[a,b]$ is a 95%-confidence interval for an unknown quantity θ if

$$Pr[\theta \in [a,b]] \geq 95\%.$$

Confidence Interval

The following definition captures precisely the notion of confidence.

Definition: Confidence Interval

An interval $[a, b]$ is a 95%-confidence interval for an unknown quantity θ if

$$Pr[\theta \in [a, b]] \geq 95\%.$$

The interval $[a, b]$ is calculated on the basis of observations.

Confidence Interval

The following definition captures precisely the notion of confidence.

Definition: Confidence Interval

An interval $[a,b]$ is a 95%-confidence interval for an unknown quantity θ if

$$Pr[\theta \in [a,b]] \geq 95\%.$$

The interval $[a,b]$ is calculated on the basis of observations.

Here is a typical framework.

Confidence Interval

The following definition captures precisely the notion of confidence.

Definition: Confidence Interval

An interval $[a, b]$ is a 95%-confidence interval for an unknown quantity θ if

$$Pr[\theta \in [a, b]] \geq 95\%.$$

The interval $[a, b]$ is calculated on the basis of observations.

Here is a typical framework. Assume that $X_1, X_2, \ldots, X_n$ are i.i.d. and have a distribution that depends on some parameter θ.

Confidence Interval

The following definition captures precisely the notion of confidence.

Definition: Confidence Interval

An interval $[a,b]$ is a 95%-confidence interval for an unknown quantity θ if

$$Pr[\theta \in [a,b]] \geq 95\%.$$

The interval $[a,b]$ is calculated on the basis of observations.

Here is a typical framework. Assume that $X_1, X_2, \ldots, X_n$ are i.i.d. and have a distribution that depends on some parameter θ.

For instance, $X_n = B(\theta)$.

Confidence Interval

The following definition captures precisely the notion of confidence.

Definition: Confidence Interval

An interval $[a,b]$ is a 95%-confidence interval for an unknown quantity θ if

$$Pr[\theta \in [a,b]] \geq 95\%.$$

The interval $[a,b]$ is calculated on the basis of observations.

Here is a typical framework. Assume that $X_1, X_2, \ldots, X_n$ are i.i.d. and have a distribution that depends on some parameter θ.

For instance, $X_n = B(\theta)$.

Thus, more precisely,

Confidence Interval

The following definition captures precisely the notion of confidence.

Definition: Confidence Interval

An interval $[a,b]$ is a 95%-confidence interval for an unknown quantity θ if

$$Pr[\theta \in [a,b]] \geq 95\%.$$

The interval $[a,b]$ is calculated on the basis of observations.

Here is a typical framework. Assume that $X_1, X_2, \ldots, X_n$ are i.i.d. and have a distribution that depends on some parameter θ.

For instance, $X_n = B(\theta)$.

Thus, more precisely, given θ, the random variables X_n are i.i.d. with a known distribution (that depends on θ).

Confidence Interval

The following definition captures precisely the notion of confidence.

Definition: Confidence Interval

An interval $[a,b]$ is a 95%-confidence interval for an unknown quantity θ if

$$Pr[\theta \in [a,b]] \geq 95\%.$$

The interval $[a,b]$ is calculated on the basis of observations.

Here is a typical framework. Assume that $X_1, X_2, \ldots, X_n$ are i.i.d. and have a distribution that depends on some parameter θ.

For instance, $X_n = B(\theta)$.

Thus, more precisely, given θ, the random variables X_n are i.i.d. with a known distribution (that depends on θ).

- We observe $X_1, \ldots, X_n$

Confidence Interval

The following definition captures precisely the notion of confidence.

Definition: Confidence Interval

An interval $[a,b]$ is a 95%-confidence interval for an unknown quantity θ if

$$Pr[\theta \in [a,b]] \geq 95\%.$$

The interval $[a,b]$ is calculated on the basis of observations.

Here is a typical framework. Assume that $X_1, X_2, \ldots, X_n$ are i.i.d. and have a distribution that depends on some parameter θ.

For instance, $X_n = B(\theta)$.

Thus, more precisely, given θ, the random variables X_n are i.i.d. with a known distribution (that depends on θ).

- We observe $X_1, \ldots, X_n$
- We calculate $a = a(X_1, \ldots, X_n)$ and $b = b(X_1, \ldots, X_n)$

Confidence Interval

The following definition captures precisely the notion of confidence.

Definition: Confidence Interval

An interval $[a, b]$ is a 95%-confidence interval for an unknown quantity θ if

$$Pr[\theta \in [a, b]] \geq 95\%.$$

The interval $[a, b]$ is calculated on the basis of observations.

Here is a typical framework. Assume that $X_1, X_2, \ldots, X_n$ are i.i.d. and have a distribution that depends on some parameter θ.

For instance, $X_n = B(\theta)$.

Thus, more precisely, given θ, the random variables X_n are i.i.d. with a known distribution (that depends on θ).

- We observe $X_1, \ldots, X_n$
- We calculate $a = a(X_1, \ldots, X_n)$ and $b = b(X_1, \ldots, X_n)$
- If we can guarantee that $Pr[\theta \in [a, b]] \geq 95\%$,

Confidence Interval

The following definition captures precisely the notion of confidence.

Definition: Confidence Interval

An interval $[a,b]$ is a 95%-confidence interval for an unknown quantity θ if

$$Pr[\theta \in [a,b]] \geq 95\%.$$

The interval $[a,b]$ is calculated on the basis of observations.

Here is a typical framework. Assume that $X_1, X_2, \ldots, X_n$ are i.i.d. and have a distribution that depends on some parameter θ.

For instance, $X_n = B(\theta)$.

Thus, more precisely, given θ, the random variables X_n are i.i.d. with a known distribution (that depends on θ).

- We observe $X_1, \ldots, X_n$
- We calculate $a = a(X_1, \ldots, X_n)$ and $b = b(X_1, \ldots, X_n)$
- If we can guarantee that $Pr[\theta \in [a,b]] \geq 95\%$, then $[a,b]$ is a 95%-CI for θ.

Confidence Interval: Applications

Confidence Interval: Applications

- We poll 1000 people.

Confidence Interval: Applications

- We poll 1000 people.
  - Among those, 48% declare they will vote for Trump.

Confidence Interval: Applications

- We poll 1000 people.
  - Among those, 48% declare they will vote for Trump.
  - We do some calculations ....

Confidence Interval: Applications

- We poll 1000 people.
  - Among those, 48% declare they will vote for Trump.
  - We do some calculations ....
  - We conclude that $[0.43, 0.53]$ is a 95%-CI for

Confidence Interval: Applications

- We poll 1000 people.
  - Among those, 48% declare they will vote for Trump.
  - We do some calculations ....
  - We conclude that $[0.43, 0.53]$ is a 95%-CI for the fraction of all the voters who will vote for Trump.

Confidence Interval: Applications

- We poll 1000 people.
  - Among those, 48% declare they will vote for Trump.
  - We do some calculations ....
  - We conclude that $[0.43, 0.53]$ is a 95%-CI for the fraction of all the voters who will vote for Trump.
- We observe $1,000$ heart valve replacements that were performed by Dr. Bill.

Confidence Interval: Applications

- We poll 1000 people.
  - Among those, 48% declare they will vote for Trump.
  - We do some calculations ....
  - We conclude that $[0.43, 0.53]$ is a 95%-CI for the fraction of all the voters who will vote for Trump.
- We observe $1,000$ heart valve replacements that were performed by Dr. Bill.
  - Among those, 35 patients had to be redone.

Confidence Interval: Applications

- We poll 1000 people.
  - Among those, 48% declare they will vote for Trump.
  - We do some calculations ....
  - We conclude that $[0.43, 0.53]$ is a 95%-CI for the fraction of all the voters who will vote for Trump.
- We observe $1,000$ heart valve replacements that were performed by Dr. Bill.
  - Among those, 35 patients had to be redone.
  - We do some calculations ...

Confidence Interval: Applications

- We poll 1000 people.
  - Among those, 48% declare they will vote for Trump.
  - We do some calculations ....
  - We conclude that $[0.43, 0.53]$ is a 95%-CI for the fraction of all the voters who will vote for Trump.
- We observe $1,000$ heart valve replacements that were performed by Dr. Bill.
  - Among those, 35 patients had to be redone.
  - We do some calculations ...
  - We conclude that $[1\%, 5\%]$ is a 95%-CI for the probability of dying during that surgery by Dr. Bill.

Confidence Interval: Applications

- We poll 1000 people.
  - Among those, 48% declare they will vote for Trump.
  - We do some calculations ....
  - We conclude that $[0.43, 0.53]$ is a 95%-CI for the fraction of all the voters who will vote for Trump.
- We observe $1,000$ heart valve replacements that were performed by Dr. Bill.
  - Among those, 35 patients had to be redone.
  - We do some calculations ...
  - We conclude that $[1\%, 5\%]$ is a 95%-CI for the probability of dying during that surgery by Dr. Bill.
  - We do a similar calculation for Dr. Fred.

Confidence Interval: Applications

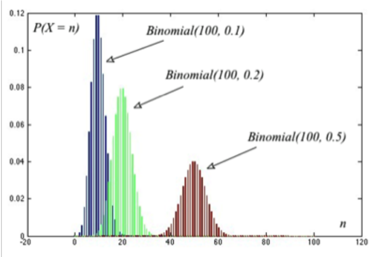
- We poll 1000 people.
  - Among those, 48% declare they will vote for Trump.
  - We do some calculations ....
  - We conclude that $[0.43, 0.53]$ is a 95%-CI for the fraction of all the voters who will vote for Trump.
- We observe $1,000$ heart valve replacements that were performed by Dr. Bill.
  - Among those, 35 patients had to be redone.
  - We do some calculations ...
  - We conclude that $[1\%, 5\%]$ is a 95%-CI for the probability of dying during that surgery by Dr. Bill.
  - We do a similar calculation for Dr. Fred.
  - We find that $[8\%, 12\%]$ is a 95%-CI for Dr. Fred's surgery.

Confidence Interval: Applications

- We poll 1000 people.
  - Among those, 48% declare they will vote for Trump.
  - We do some calculations ....
  - We conclude that $[0.43, 0.53]$ is a 95%-CI for the fraction of all the voters who will vote for Trump.
- We observe $1,000$ heart valve replacements that were performed by Dr. Bill.
  - Among those, 35 patients had to be redone.
  - We do some calculations ...
  - We conclude that $[1\%, 5\%]$ is a 95%-CI for the probability of dying during that surgery by Dr. Bill.
  - We do a similar calculation for Dr. Fred.
  - We find that $[8\%, 12\%]$ is a 95%-CI for Dr. Fred's surgery.
  - What surgeon do you choose?

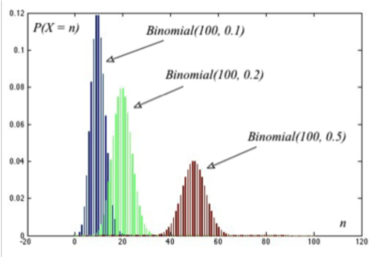
Coin Flips: Intuition

Coin Flips: Intuition

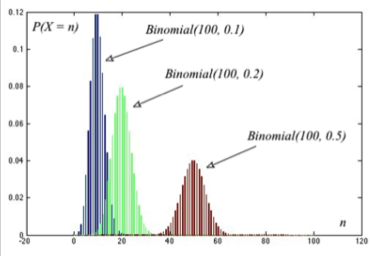


Coin Flips: Intuition

Say that you flip a coin $n = 100$ times and observe 20 Hs.



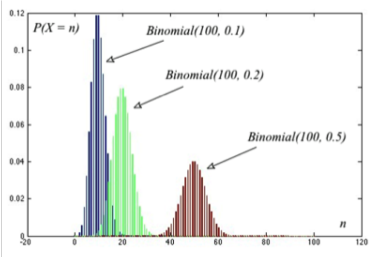
Coin Flips: Intuition



Say that you flip a coin $n = 100$ times and observe 20 Hs.

If $p := Pr[H] = 0.5$, this event is very unlikely.

Coin Flips: Intuition

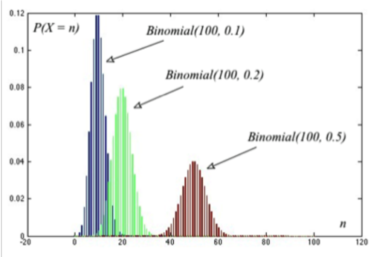


Say that you flip a coin $n = 100$ times and observe 20 Hs.

If $p := Pr[H] = 0.5$, this event is very unlikely.

Intuitively, if is unlikely that the fraction of Hs, say A_n,

Coin Flips: Intuition

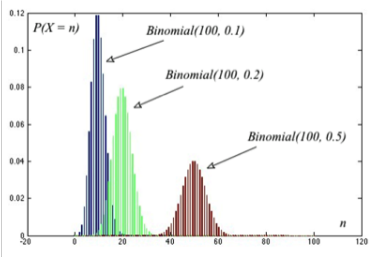


Say that you flip a coin $n = 100$ times and observe 20 Hs.

If $p := Pr[H] = 0.5$, this event is very unlikely.

Intuitively, if is unlikely that the fraction of Hs, say A_n, differs a lot from $p := Pr[H]$.

Coin Flips: Intuition



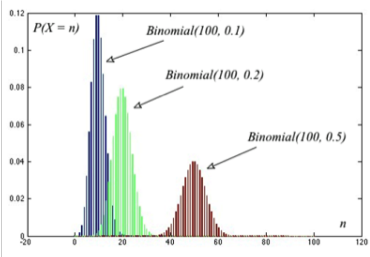
Say that you flip a coin $n = 100$ times and observe 20 Hs.

If $p := Pr[H] = 0.5$, this event is very unlikely.

Intuitively, if is unlikely that the fraction of Hs, say A_n, differs a lot from $p := Pr[H]$.

Thus, it is unlikely that p differs a lot from A_n.

Coin Flips: Intuition



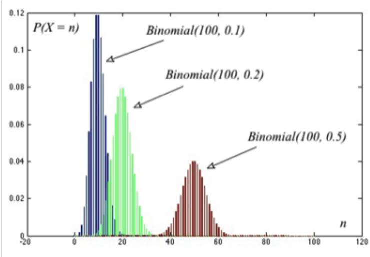
Say that you flip a coin $n = 100$ times and observe 20 Hs.

If $p := Pr[H] = 0.5$, this event is very unlikely.

Intuitively, if is unlikely that the fraction of Hs, say A_n, differs a lot from $p := Pr[H]$.

Thus, it is unlikely that p differs a lot from A_n. Hence, one should be able to build a confidence interval $[A_n - \varepsilon, A_n + \varepsilon]$ for p.

Coin Flips: Intuition



Say that you flip a coin $n = 100$ times and observe 20 Hs.

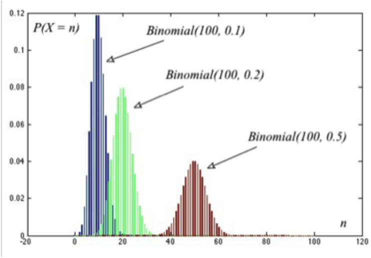
If $p := Pr[H] = 0.5$, this event is very unlikely.

Intuitively, if is unlikely that the fraction of Hs, say A_n, differs a lot from $p := Pr[H]$.

Thus, it is unlikely that p differs a lot from A_n. Hence, one should be able to build a confidence interval $[A_n - \varepsilon, A_n + \varepsilon]$ for p.

The key idea is that $|A_n - p| \leq \varepsilon \Leftrightarrow p \in [A_n - \varepsilon, A_n + \varepsilon]$.

Coin Flips: Intuition



Say that you flip a coin $n = 100$ times and observe 20 Hs.

If $p := Pr[H] = 0.5$, this event is very unlikely.

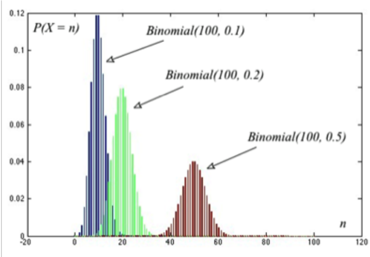
Intuitively, if is unlikely that the fraction of Hs, say A_n, differs a lot from $p := Pr[H]$.

Thus, it is unlikely that p differs a lot from A_n. Hence, one should be able to build a confidence interval $[A_n - \varepsilon, A_n + \varepsilon]$ for p.

The key idea is that $|A_n - p| \leq \varepsilon \Leftrightarrow p \in [A_n - \varepsilon, A_n + \varepsilon]$.

Thus, $Pr[|A_n - p| > \varepsilon] \leq 5\% \Leftrightarrow Pr[p \in [A_n - \varepsilon, A_n + \varepsilon]] \geq 95\%$.

Coin Flips: Intuition



Say that you flip a coin $n = 100$ times and observe 20 Hs.

If $p := Pr[H] = 0.5$, this event is very unlikely.

Intuitively, if is unlikely that the fraction of Hs, say A_n, differs a lot from $p := Pr[H]$.

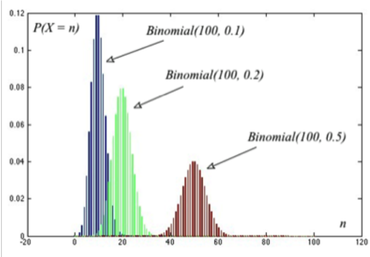
Thus, it is unlikely that p differs a lot from A_n. Hence, one should be able to build a confidence interval $[A_n - \varepsilon, A_n + \varepsilon]$ for p.

The key idea is that $|A_n - p| \leq \varepsilon \Leftrightarrow p \in [A_n - \varepsilon, A_n + \varepsilon]$.

Thus, $Pr[|A_n - p| > \varepsilon] \leq 5\% \Leftrightarrow Pr[p \in [A_n - \varepsilon, A_n + \varepsilon]] \geq 95\%$.

It remains to find ε such that $Pr[|A_n - p| > \varepsilon] \leq 5\%$.

Coin Flips: Intuition



Say that you flip a coin $n = 100$ times and observe 20 Hs.

If $p := Pr[H] = 0.5$, this event is very unlikely.

Intuitively, if is unlikely that the fraction of Hs, say A_n, differs a lot from $p := Pr[H]$.

Thus, it is unlikely that p differs a lot from A_n. Hence, one should be able to build a confidence interval $[A_n - \varepsilon, A_n + \varepsilon]$ for p.

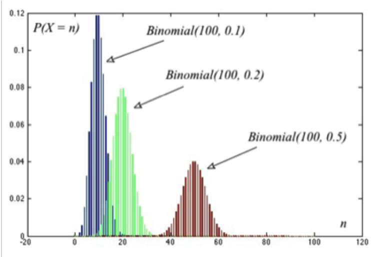
The key idea is that $|A_n - p| \leq \varepsilon \Leftrightarrow p \in [A_n - \varepsilon, A_n + \varepsilon]$.

Thus, $Pr[|A_n - p| > \varepsilon] \leq 5\% \Leftrightarrow Pr[p \in [A_n - \varepsilon, A_n + \varepsilon]] \geq 95\%$.

It remains to find ε such that $Pr[|A_n - p| > \varepsilon] \leq 5\%$.

One approach:

Coin Flips: Intuition



Say that you flip a coin $n = 100$ times and observe 20 Hs.

If $p := Pr[H] = 0.5$, this event is very unlikely.

Intuitively, if is unlikely that the fraction of Hs, say A_n, differs a lot from $p := Pr[H]$.

Thus, it is unlikely that p differs a lot from A_n. Hence, one should be able to build a confidence interval $[A_n - \varepsilon, A_n + \varepsilon]$ for p.

The key idea is that $|A_n - p| \leq \varepsilon \Leftrightarrow p \in [A_n - \varepsilon, A_n + \varepsilon]$.

Thus, $Pr[|A_n - p| > \varepsilon] \leq 5\% \Leftrightarrow Pr[p \in [A_n - \varepsilon, A_n + \varepsilon]] \geq 95\%$.

It remains to find ε such that $Pr[|A_n - p| > \varepsilon] \leq 5\%$.

One approach: Chebyshev.

Confidence Interval with Chebyshev

Confidence Interval with Chebyshev

- Flip a coin n times.

Confidence Interval with Chebyshev

- Flip a coin n times. Let A_n be the fraction of Hs.

Confidence Interval with Chebyshev

- Flip a coin n times. Let A_n be the fraction of Hs.
- Can we find ε such that $Pr[|A_n - p| > \varepsilon] \leq 5\%$?

Confidence Interval with Chebyshev

- Flip a coin n times. Let A_n be the fraction of Hs.
- Can we find ε such that $Pr[|A_n - p| > \varepsilon] \leq 5\%$?

Using Chebyshev, we will see that $\varepsilon = 2.25 \frac{1}{\sqrt{n}}$ works.

Confidence Interval with Chebyshev

- Flip a coin n times. Let A_n be the fraction of Hs.
- Can we find ε such that $Pr[|A_n - p| > \varepsilon] \leq 5\%$?

Using Chebyshev, we will see that $\varepsilon = 2.25\frac{1}{\sqrt{n}}$ works. Thus

$$[A_n - \frac{2.25}{\sqrt{n}}, A_n + \frac{2.25}{\sqrt{n}}] \text{ is a 95\%-CI for } p.$$

Confidence Interval with Chebyshev

- Flip a coin n times. Let A_n be the fraction of Hs.
- Can we find ε such that $Pr[|A_n - p| > \varepsilon] \leq 5\%$?

Using Chebyshev, we will see that $\varepsilon = 2.25\frac{1}{\sqrt{n}}$ works. Thus

$$[A_n - \frac{2.25}{\sqrt{n}}, A_n + \frac{2.25}{\sqrt{n}}] \text{ is a 95\%-CI for } p.$$

Example: If $n = 1500$, then $Pr[p \in [A_n - 0.05, A_n + 0.05]] \geq 95\%$.

Confidence Interval with Chebyshev

- Flip a coin n times. Let A_n be the fraction of Hs.
- Can we find ε such that $Pr[|A_n - p| > \varepsilon] \leq 5\%$?

Using Chebyshev, we will see that $\varepsilon = 2.25\frac{1}{\sqrt{n}}$ works. Thus

$$[A_n - \frac{2.25}{\sqrt{n}}, A_n + \frac{2.25}{\sqrt{n}}] \text{ is a 95\%-CI for } p.$$

Example: If $n = 1500$, then $Pr[p \in [A_n - 0.05, A_n + 0.05]] \geq 95\%$.

In fact, $a = \frac{1}{\sqrt{n}}$ works,

Confidence Interval with Chebyshev

- Flip a coin n times. Let A_n be the fraction of Hs.
- Can we find ε such that $Pr[|A_n - p| > \varepsilon] \leq 5\%$?

Using Chebyshev, we will see that $\varepsilon = 2.25\frac{1}{\sqrt{n}}$ works. Thus

$$[A_n - \frac{2.25}{\sqrt{n}}, A_n + \frac{2.25}{\sqrt{n}}] \text{ is a 95\%-CI for } p.$$

Example: If $n = 1500$, then $Pr[p \in [A_n - 0.05, A_n + 0.05]] \geq 95\%$.

In fact, $a = \frac{1}{\sqrt{n}}$ works, so that with $n = 1,500$ one has
$Pr[p \in [A_n - 0.02, A_n + 0.02]] \geq 95\%$.

Confidence Intervals: Result

Confidence Intervals: Result

Theorem:
Let X_n be i.i.d. with mean μ and variance σ^2.

Confidence Intervals: Result

Theorem:
Let X_n be i.i.d. with mean μ and variance σ^2.
Define $A_n = \frac{X_1+\cdots+X_n}{n}$.

Confidence Intervals: Result

Theorem:
Let X_n be i.i.d. with mean μ and variance σ^2.
Define $A_n = \frac{X_1+\cdots+X_n}{n}$. Then,

$$Pr[\mu \in [A_n - 4.5\frac{\sigma}{\sqrt{n}}, A_n + 4.5\frac{\sigma}{\sqrt{n}}]] \geq 95\%.$$

Confidence Intervals: Result

Theorem:
Let X_n be i.i.d. with mean μ and variance σ^2.
Define $A_n = \frac{X_1 + \cdots + X_n}{n}$. Then,

$$Pr[\mu \in [A_n - 4.5\frac{\sigma}{\sqrt{n}}, A_n + 4.5\frac{\sigma}{\sqrt{n}}]] \geq 95\%.$$

Thus, $[A_n - 4.5\frac{\sigma}{\sqrt{n}}, A_n + 4.5\frac{\sigma}{\sqrt{n}}]$] is a 95%-CI for μ.

Confidence Intervals: Result

Theorem:
Let X_n be i.i.d. with mean μ and variance σ^2.
Define $A_n = \frac{X_1 + \cdots + X_n}{n}$. Then,

$$Pr[\mu \in [A_n - 4.5\frac{\sigma}{\sqrt{n}}, A_n + 4.5\frac{\sigma}{\sqrt{n}}]] \geq 95\%.$$

Thus, $[A_n - 4.5\frac{\sigma}{\sqrt{n}}, A_n + 4.5\frac{\sigma}{\sqrt{n}}]]$ is a 95%-CI for μ.

Example:

Confidence Intervals: Result

Theorem:
Let X_n be i.i.d. with mean μ and variance σ^2.
Define $A_n = \frac{X_1 + \cdots + X_n}{n}$. Then,

$$Pr[\mu \in [A_n - 4.5\frac{\sigma}{\sqrt{n}}, A_n + 4.5\frac{\sigma}{\sqrt{n}}]] \geq 95\%.$$

Thus, $[A_n - 4.5\frac{\sigma}{\sqrt{n}}, A_n + 4.5\frac{\sigma}{\sqrt{n}}]$] is a 95%-CI for μ.

Example: Let $X_n = 1\{$ coin n yields $H\}$.

Confidence Intervals: Result

Theorem:
Let X_n be i.i.d. with mean μ and variance σ^2.
Define $A_n = \frac{X_1 + \cdots + X_n}{n}$. Then,

$$Pr[\mu \in [A_n - 4.5\frac{\sigma}{\sqrt{n}}, A_n + 4.5\frac{\sigma}{\sqrt{n}}]] \geq 95\%.$$

Thus, $[A_n - 4.5\frac{\sigma}{\sqrt{n}}, A_n + 4.5\frac{\sigma}{\sqrt{n}}]$] is a 95%-CI for μ.

Example: Let $X_n = 1\{$ coin n yields $H\}$. Then

$$\mu = E[X_n] = p := Pr[H].$$

Confidence Intervals: Result

Theorem:
Let X_n be i.i.d. with mean μ and variance σ^2.
Define $A_n = \frac{X_1 + \cdots + X_n}{n}$. Then,

$$Pr[\mu \in [A_n - 4.5\frac{\sigma}{\sqrt{n}}, A_n + 4.5\frac{\sigma}{\sqrt{n}}]] \geq 95\%.$$

Thus, $[A_n - 4.5\frac{\sigma}{\sqrt{n}}, A_n + 4.5\frac{\sigma}{\sqrt{n}}]]$ is a 95%-CI for μ.

Example: Let $X_n = 1\{$ coin n yields $H\}$. Then

$$\mu = E[X_n] = p := Pr[H]. \text{ Also, } \sigma^2 = var(X_n) =$$

Confidence Intervals: Result

Theorem:
Let X_n be i.i.d. with mean μ and variance σ^2.
Define $A_n = \frac{X_1 + \cdots + X_n}{n}$. Then,

$$Pr[\mu \in [A_n - 4.5\frac{\sigma}{\sqrt{n}}, A_n + 4.5\frac{\sigma}{\sqrt{n}}]] \geq 95\%.$$

Thus, $[A_n - 4.5\frac{\sigma}{\sqrt{n}}, A_n + 4.5\frac{\sigma}{\sqrt{n}}]]$ is a 95%-CI for μ.

Example: Let $X_n = 1\{$ coin n yields $H\}$. Then

$$\mu = E[X_n] = p := Pr[H]. \text{ Also, } \sigma^2 = var(X_n) = p(1-p)$$

Confidence Intervals: Result

Theorem:
Let X_n be i.i.d. with mean μ and variance σ^2.
Define $A_n = \frac{X_1 + \cdots + X_n}{n}$. Then,

$$Pr[\mu \in [A_n - 4.5\frac{\sigma}{\sqrt{n}}, A_n + 4.5\frac{\sigma}{\sqrt{n}}]] \geq 95\%.$$

Thus, $[A_n - 4.5\frac{\sigma}{\sqrt{n}}, A_n + 4.5\frac{\sigma}{\sqrt{n}}]]$ is a 95%-CI for μ.

Example: Let $X_n = 1\{$ coin n yields $H\}$. Then

$$\mu = E[X_n] = p := Pr[H]. \text{ Also, } \sigma^2 = var(X_n) = p(1-p) \leq \frac{1}{4}.$$

Confidence Intervals: Result

Theorem:
Let X_n be i.i.d. with mean μ and variance σ^2.
Define $A_n = \frac{X_1 + \cdots + X_n}{n}$. Then,

$$Pr[\mu \in [A_n - 4.5\frac{\sigma}{\sqrt{n}}, A_n + 4.5\frac{\sigma}{\sqrt{n}}]] \geq 95\%.$$

Thus, $[A_n - 4.5\frac{\sigma}{\sqrt{n}}, A_n + 4.5\frac{\sigma}{\sqrt{n}}]$] is a 95%-CI for μ.

Example: Let $X_n = 1\{$ coin n yields $H\}$. Then

$$\mu = E[X_n] = p := Pr[H]. \text{ Also, } \sigma^2 = var(X_n) = p(1-p) \leq \frac{1}{4}.$$

Hence, $[A_n - 4.5\frac{1/2}{\sqrt{n}}, A_n + 4.5\frac{1/2}{\sqrt{n}}]$] is a 95%-CI for p.

Confidence Interval: Analysis

Confidence Interval: Analysis

We prove the theorem, i.e., that $A_n \pm 4.5\sigma/\sqrt{n}$ is a 95%-CI for μ.

Confidence Interval: Analysis

We prove the theorem, i.e., that $A_n \pm 4.5\sigma/\sqrt{n}$ is a 95%-CI for μ.

From Chebyshev:

$$Pr[|A_n - \mu| \geq 4.5\sigma/\sqrt{n}] \leq \frac{var(A_n)}{[4.5\sigma/\sqrt{n}]^2}$$

Confidence Interval: Analysis

We prove the theorem, i.e., that $A_n \pm 4.5\sigma/\sqrt{n}$ is a 95%-CI for μ.

From Chebyshev:

$$Pr[|A_n - \mu| \geq 4.5\sigma/\sqrt{n}] \leq \frac{var(A_n)}{[4.5\sigma/\sqrt{n}]^2} = \frac{n}{20\sigma^2} var(A_n).$$

Confidence Interval: Analysis

We prove the theorem, i.e., that $A_n \pm 4.5\sigma/\sqrt{n}$ is a 95%-CI for μ.

From Chebyshev:

$$Pr[|A_n - \mu| \geq 4.5\sigma/\sqrt{n}] \leq \frac{var(A_n)}{[4.5\sigma/\sqrt{n}]^2} = \frac{n}{20\sigma^2} var(A_n).$$

Now,

$$var(A_n) \quad = \quad var(\frac{X_1 + \cdots + X_n}{n}) =$$

Confidence Interval: Analysis

We prove the theorem, i.e., that $A_n \pm 4.5\sigma/\sqrt{n}$ is a 95%-CI for μ.

From Chebyshev:

$$Pr[|A_n - \mu| \geq 4.5\sigma/\sqrt{n}] \leq \frac{var(A_n)}{[4.5\sigma/\sqrt{n}]^2} = \frac{n}{20\sigma^2} var(A_n).$$

Now,

$$var(A_n) = var(\frac{X_1 + \cdots + X_n}{n}) = \frac{1}{n^2} var(X_1 + \cdots + X_n)$$

Confidence Interval: Analysis

We prove the theorem, i.e., that $A_n \pm 4.5\sigma/\sqrt{n}$ is a 95%-CI for μ.

From Chebyshev:

$$Pr[|A_n - \mu| \geq 4.5\sigma/\sqrt{n}] \leq \frac{var(A_n)}{[4.5\sigma/\sqrt{n}]^2} = \frac{n}{20\sigma^2} var(A_n).$$

Now,

$$\begin{aligned} var(A_n) &= var(\frac{X_1 + \cdots + X_n}{n}) = \frac{1}{n^2} var(X_1 + \cdots + X_n) \\ &= \frac{1}{n^2} \times n.var(X_1) = \end{aligned}$$

Confidence Interval: Analysis

We prove the theorem, i.e., that $A_n \pm 4.5\sigma/\sqrt{n}$ is a 95%-CI for μ.

From Chebyshev:

$$Pr[|A_n - \mu| \geq 4.5\sigma/\sqrt{n}] \leq \frac{var(A_n)}{[4.5\sigma/\sqrt{n}]^2} = \frac{n}{20\sigma^2} var(A_n).$$

Now,

$$\begin{aligned} var(A_n) &= var(\frac{X_1 + \cdots + X_n}{n}) = \frac{1}{n^2} var(X_1 + \cdots + X_n) \\ &= \frac{1}{n^2} \times n.var(X_1) = \frac{1}{n}\sigma^2. \end{aligned}$$

Confidence Interval: Analysis

We prove the theorem, i.e., that $A_n \pm 4.5\sigma/\sqrt{n}$ is a 95%-CI for μ.

From Chebyshev:

$$Pr[|A_n - \mu| \geq 4.5\sigma/\sqrt{n}] \leq \frac{var(A_n)}{[4.5\sigma/\sqrt{n}]^2} = \frac{n}{20\sigma^2} var(A_n).$$

Now,

$$\begin{aligned} var(A_n) &= var(\frac{X_1 + \cdots + X_n}{n}) = \frac{1}{n^2} var(X_1 + \cdots + X_n) \\ &= \frac{1}{n^2} \times n.var(X_1) = \frac{1}{n}\sigma^2. \end{aligned}$$

Hence,

$$Pr[|A_n - \mu| \geq 4.5\sigma/\sqrt{n}] \leq \frac{n}{20\sigma^2} \times \frac{1}{n}\sigma^2$$

Confidence Interval: Analysis

We prove the theorem, i.e., that $A_n \pm 4.5\sigma/\sqrt{n}$ is a 95%-CI for μ.

From Chebyshev:

$$Pr[|A_n - \mu| \geq 4.5\sigma/\sqrt{n}] \leq \frac{var(A_n)}{[4.5\sigma/\sqrt{n}]^2} = \frac{n}{20\sigma^2} var(A_n).$$

Now,

$$\begin{aligned} var(A_n) &= var(\frac{X_1 + \cdots + X_n}{n}) = \frac{1}{n^2} var(X_1 + \cdots + X_n) \\ &= \frac{1}{n^2} \times n.var(X_1) = \frac{1}{n}\sigma^2. \end{aligned}$$

Hence,

$$Pr[|A_n - \mu| \geq 4.5\sigma/\sqrt{n}] \leq \frac{n}{20\sigma^2} \times \frac{1}{n}\sigma^2 = 5\%.$$

Confidence Interval: Analysis

We prove the theorem, i.e., that $A_n \pm 4.5\sigma/\sqrt{n}$ is a 95%-CI for μ.

From Chebyshev:

$$Pr[|A_n - \mu| \geq 4.5\sigma/\sqrt{n}] \leq \frac{var(A_n)}{[4.5\sigma/\sqrt{n}]^2} = \frac{n}{20\sigma^2} var(A_n).$$

Now,

$$\begin{aligned} var(A_n) &= var(\frac{X_1 + \cdots + X_n}{n}) = \frac{1}{n^2} var(X_1 + \cdots + X_n) \\ &= \frac{1}{n^2} \times n.var(X_1) = \frac{1}{n}\sigma^2. \end{aligned}$$

Hence,

$$Pr[|A_n - \mu| \geq 4.5\sigma/\sqrt{n}] \leq \frac{n}{20\sigma^2} \times \frac{1}{n}\sigma^2 = 5\%.$$

Thus,

$$Pr[|A_n - \mu| \leq 4.5\sigma/\sqrt{n}] \geq 95\%.$$

Confidence Interval: Analysis

We prove the theorem, i.e., that $A_n \pm 4.5\sigma/\sqrt{n}$ is a 95%-CI for μ.

From Chebyshev:

$$Pr[|A_n - \mu| \geq 4.5\sigma/\sqrt{n}] \leq \frac{var(A_n)}{[4.5\sigma/\sqrt{n}]^2} = \frac{n}{20\sigma^2} var(A_n).$$

Now,

$$\begin{aligned} var(A_n) &= var(\frac{X_1 + \cdots + X_n}{n}) = \frac{1}{n^2} var(X_1 + \cdots + X_n) \\ &= \frac{1}{n^2} \times n.var(X_1) = \frac{1}{n}\sigma^2. \end{aligned}$$

Hence,

$$Pr[|A_n - \mu| \geq 4.5\sigma/\sqrt{n}] \leq \frac{n}{20\sigma^2} \times \frac{1}{n}\sigma^2 = 5\%.$$

Thus,

$$Pr[|A_n - \mu| \leq 4.5\sigma/\sqrt{n}] \geq 95\%.$$

Hence,

$$Pr[\mu \in [A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}]] \geq 95\%.$$

$\square$

Confidence interval for p in $B(p)$

Confidence interval for p in $B(p)$

Let X_n be i.i.d. $B(p)$.

Confidence interval for p in $B(p)$

Let X_n be i.i.d. $B(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

Confidence interval for p in $B(p)$

Let X_n be i.i.d. $B(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

Theorem:

$$[A_n - \frac{2.25}{\sqrt{n}}, A_n + \frac{2.25}{\sqrt{n}}] \text{ is a 95\%-CI for } p.$$

Confidence interval for p in $B(p)$

Let X_n be i.i.d. $B(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

Theorem:

$$[A_n - \frac{2.25}{\sqrt{n}}, A_n + \frac{2.25}{\sqrt{n}}] \text{ is a 95\%-CI for } p.$$

Proof:

Confidence interval for p in $B(p)$

Let X_n be i.i.d. $B(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

Theorem:

$$[A_n - \frac{2.25}{\sqrt{n}}, A_n + \frac{2.25}{\sqrt{n}}] \text{ is a 95\%-CI for } p.$$

Proof:

We have just seen that

$$Pr[\mu \in [A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}]] \geq 95\%.$$

Confidence interval for p in $B(p)$

Let X_n be i.i.d. $B(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

Theorem:

$$[A_n - \frac{2.25}{\sqrt{n}}, A_n + \frac{2.25}{\sqrt{n}}] \text{ is a 95\%-CI for } p.$$

Proof:

We have just seen that

$$Pr[\mu \in [A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}]] \geq 95\%.$$

Here, $\mu = p$

Confidence interval for p in $B(p)$

Let X_n be i.i.d. $B(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

Theorem:

$$[A_n - \frac{2.25}{\sqrt{n}}, A_n + \frac{2.25}{\sqrt{n}}] \text{ is a 95\%-CI for } p.$$

Proof:

We have just seen that

$$Pr[\mu \in [A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}]] \geq 95\%.$$

Here, $\mu = p$ and $\sigma^2 = p(1-p)$.

Confidence interval for p in $B(p)$

Let X_n be i.i.d. $B(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

Theorem:

$$[A_n - \frac{2.25}{\sqrt{n}}, A_n + \frac{2.25}{\sqrt{n}}] \text{ is a 95\%-CI for } p.$$

Proof:

We have just seen that

$$Pr[\mu \in [A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}]] \geq 95\%.$$

Here, $\mu = p$ and $\sigma^2 = p(1-p)$. Thus, $\sigma^2 \leq \frac{1}{4}$

Confidence interval for p in $B(p)$

Let X_n be i.i.d. $B(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

Theorem:

$$[A_n - \frac{2.25}{\sqrt{n}}, A_n + \frac{2.25}{\sqrt{n}}] \text{ is a 95\%-CI for } p.$$

Proof:

We have just seen that

$$Pr[\mu \in [A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}]] \geq 95\%.$$

Here, $\mu = p$ and $\sigma^2 = p(1-p)$. Thus, $\sigma^2 \leq \frac{1}{4}$ and $\sigma \leq \frac{1}{2}$.

Confidence interval for p in $B(p)$

Let X_n be i.i.d. $B(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

Theorem:

$$[A_n - \frac{2.25}{\sqrt{n}}, A_n + \frac{2.25}{\sqrt{n}}] \text{ is a 95\%-CI for } p.$$

Proof:

We have just seen that

$$Pr[\mu \in [A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}]] \geq 95\%.$$

Here, $\mu = p$ and $\sigma^2 = p(1-p)$. Thus, $\sigma^2 \leq \frac{1}{4}$ and $\sigma \leq \frac{1}{2}$.

Thus,

$$Pr[\mu \in [A_n - 4.5 \times 0.5/\sqrt{n}, A_n + 4.5 \times 0.5/\sqrt{n}]] \geq 95\%.$$

Confidence interval for p in $B(p)$

Let X_n be i.i.d. $B(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

Theorem:

$$[A_n - \frac{2.25}{\sqrt{n}}, A_n + \frac{2.25}{\sqrt{n}}] \text{ is a 95\%-CI for } p.$$

Proof:

We have just seen that

$$Pr[\mu \in [A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}]] \geq 95\%.$$

Here, $\mu = p$ and $\sigma^2 = p(1-p)$. Thus, $\sigma^2 \leq \frac{1}{4}$ and $\sigma \leq \frac{1}{2}$.

Thus,

$$Pr[\mu \in [A_n - 4.5 \times 0.5/\sqrt{n}, A_n + 4.5 \times 0.5/\sqrt{n}]] \geq 95\%.$$

$\square$

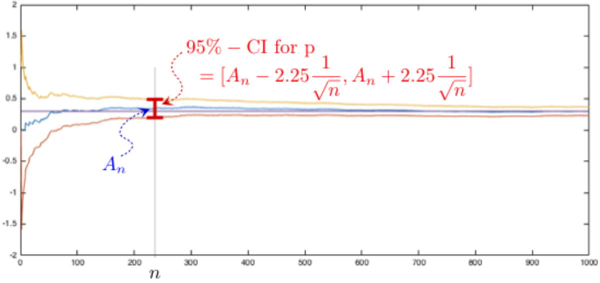
Confidence interval for p in $B(p)$

Confidence interval for p in $B(p)$

An illustration:

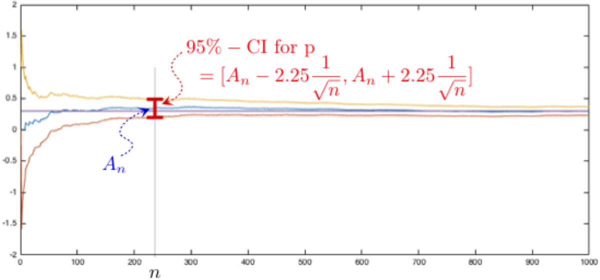
Confidence interval for p in $B(p)$

An illustration:



Confidence interval for p in $B(p)$

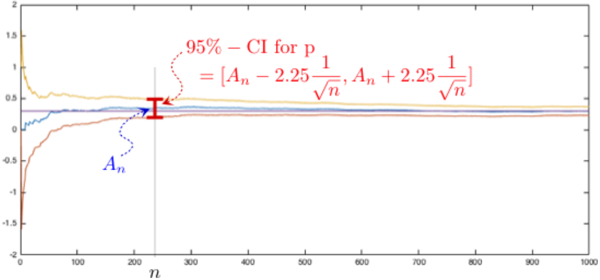
An illustration:



Good practice: You run your simulation, or experiment.

Confidence interval for p in $B(p)$

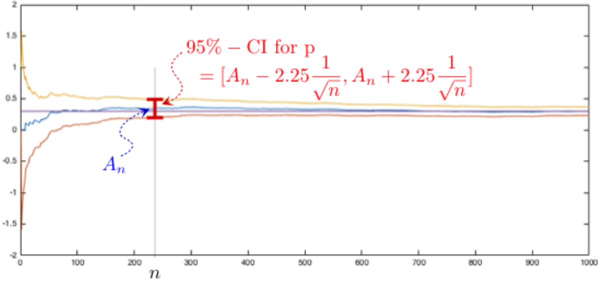
An illustration:



Good practice: You run your simulation, or experiment. You get an estimate.

Confidence interval for p in $B(p)$

An illustration:



Good practice: You run your simulation, or experiment. You get an estimate. You indicate your confidence interval.

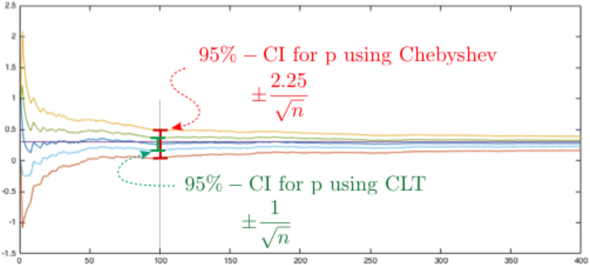
Confidence interval for p in $B(p)$

Confidence interval for p in $B(p)$

Improved CI:

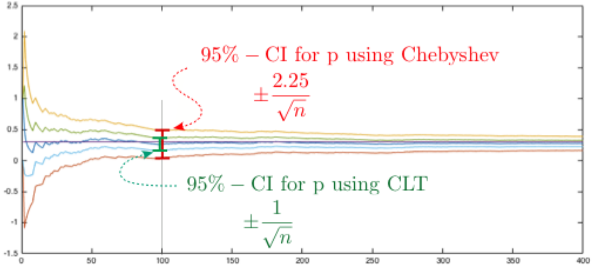
Confidence interval for p in $B(p)$

Improved CI: In fact, one can replace 2.25 by 1.



Confidence interval for p in $B(p)$

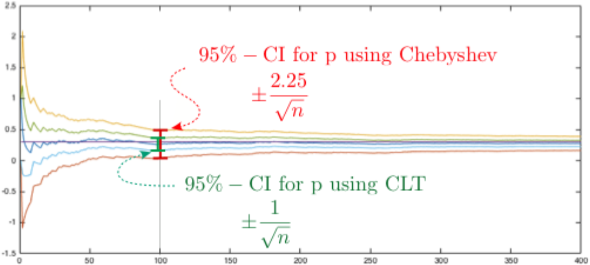
Improved CI: In fact, one can replace 2.25 by 1.



Quite a bit of work to get there:

Confidence interval for p in $B(p)$

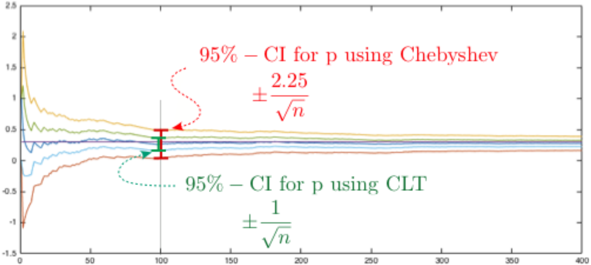
Improved CI: In fact, one can replace 2.25 by 1.



Quite a bit of work to get there: continuous random variables;

Confidence interval for p in $B(p)$

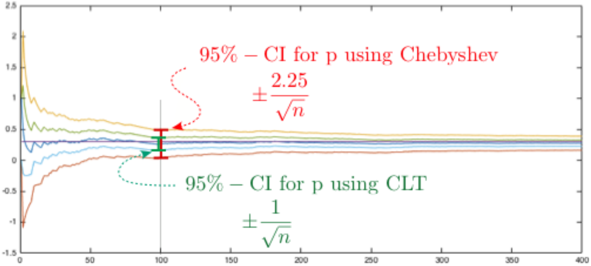
Improved CI: In fact, one can replace 2.25 by 1.



Quite a bit of work to get there: continuous random variables; Gaussian;

Confidence interval for p in $B(p)$

Improved CI: In fact, one can replace 2.25 by 1.



Quite a bit of work to get there: continuous random variables; Gaussian; Central Limit Theorem.

Confidence Interval for $1/p$ in $G(p)$

Confidence Interval for $1/p$ in $G(p)$

Let X_n be i.i.d. $G(p)$.

Confidence Interval for $1/p$ in $G(p)$

Let X_n be i.i.d. $G(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

Confidence Interval for $1/p$ in $G(p)$

Let X_n be i.i.d. $G(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

Theorem:

$$[\frac{A_n}{1 + 4.5/\sqrt{n}}, \frac{A_n}{1 - 4.5/\sqrt{n}}] \text{ is a 95\%-CI for } \frac{1}{p}.$$

Confidence Interval for $1/p$ in $G(p)$

Let X_n be i.i.d. $G(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

Theorem:

$$[\frac{A_n}{1 + 4.5/\sqrt{n}}, \frac{A_n}{1 - 4.5/\sqrt{n}}] \text{ is a 95\%-CI for } \frac{1}{p}.$$

Proof:

Confidence Interval for $1/p$ in $G(p)$

Let X_n be i.i.d. $G(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

Theorem:

$$[\frac{A_n}{1+4.5/\sqrt{n}}, \frac{A_n}{1-4.5/\sqrt{n}}] \text{ is a 95\%-CI for } \frac{1}{p}.$$

Proof: We know that

$$Pr[\mu \in [A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}]] \geq 95\%.$$

Confidence Interval for $1/p$ in $G(p)$

Let X_n be i.i.d. $G(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

Theorem:

$$[\frac{A_n}{1 + 4.5/\sqrt{n}}, \frac{A_n}{1 - 4.5/\sqrt{n}}] \text{ is a 95\%-CI for } \frac{1}{p}.$$

Proof: We know that

$$Pr[\mu \in [A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}]] \geq 95\%.$$

Here, $\mu = \frac{1}{p}$ and $\sigma = \frac{\sqrt{1-p}}{p} \leq \frac{1}{p}$.

Confidence Interval for $1/p$ in $G(p)$

Let X_n be i.i.d. $G(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

Theorem:

$$[\frac{A_n}{1+4.5/\sqrt{n}}, \frac{A_n}{1-4.5/\sqrt{n}}] \text{ is a 95\%-CI for } \frac{1}{p}.$$

Proof: We know that

$$Pr[\mu \in [A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}]] \geq 95\%.$$

Here, $\mu = \frac{1}{p}$ and $\sigma = \frac{\sqrt{1-p}}{p} \leq \frac{1}{p}$. Hence,

$$Pr[\frac{1}{p} \in [A_n - 4.5\frac{1}{p\sqrt{n}}, A_n + 4.5\frac{1}{p\sqrt{n}}]] \geq 95\%.$$

Confidence Interval for $1/p$ in $G(p)$

Let X_n be i.i.d. $G(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

Theorem:

$$[\frac{A_n}{1+4.5/\sqrt{n}}, \frac{A_n}{1-4.5/\sqrt{n}}] \text{ is a 95\%-CI for } \frac{1}{p}.$$

Proof: We know that

$$Pr[\mu \in [A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}]] \geq 95\%.$$

Here, $\mu = \frac{1}{p}$ and $\sigma = \frac{\sqrt{1-p}}{p} \leq \frac{1}{p}$. Hence,

$$Pr[\frac{1}{p} \in [A_n - 4.5\frac{1}{p\sqrt{n}}, A_n + 4.5\frac{1}{p\sqrt{n}}]] \geq 95\%.$$

Now, $A_n - 4.5\frac{1}{p\sqrt{n}} \leq \frac{1}{p} \leq \frac{1}{p} \leq A_n + 4.5\frac{1}{p\sqrt{n}}$ is equivalent to

Confidence Interval for $1/p$ in $G(p)$

Let X_n be i.i.d. $G(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

Theorem:

$$[\frac{A_n}{1+4.5/\sqrt{n}}, \frac{A_n}{1-4.5/\sqrt{n}}] \text{ is a 95\%-CI for } \frac{1}{p}.$$

Proof: We know that

$$Pr[\mu \in [A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}]] \geq 95\%.$$

Here, $\mu = \frac{1}{p}$ and $\sigma = \frac{\sqrt{1-p}}{p} \leq \frac{1}{p}$. Hence,

$$Pr[\frac{1}{p} \in [A_n - 4.5\frac{1}{p\sqrt{n}}, A_n + 4.5\frac{1}{p\sqrt{n}}]] \geq 95\%.$$

Now, $A_n - 4.5\frac{1}{p\sqrt{n}} \leq \frac{1}{p} \leq \frac{1}{p} \leq A_n + 4.5\frac{1}{p\sqrt{n}}$ is equivalent to

$$\frac{A_n}{1+4.5/\sqrt{n}} \leq \frac{1}{p} \leq \frac{A_n}{1-4.5/\sqrt{n}}.$$

Confidence Interval for $1/p$ in $G(p)$

Let X_n be i.i.d. $G(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

Theorem:

$$[\frac{A_n}{1+4.5/\sqrt{n}}, \frac{A_n}{1-4.5/\sqrt{n}}] \text{ is a 95\%-CI for } \frac{1}{p}.$$

Proof: We know that

$$Pr[\mu \in [A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}]] \geq 95\%.$$

Here, $\mu = \frac{1}{p}$ and $\sigma = \frac{\sqrt{1-p}}{p} \leq \frac{1}{p}$. Hence,

$$Pr[\frac{1}{p} \in [A_n - 4.5\frac{1}{p\sqrt{n}}, A_n + 4.5\frac{1}{p\sqrt{n}}]] \geq 95\%.$$

Now, $A_n - 4.5\frac{1}{p\sqrt{n}} \leq \frac{1}{p} \leq \frac{1}{p} \leq A_n + 4.5\frac{1}{p\sqrt{n}}$ is equivalent to

$$\frac{A_n}{1+4.5/\sqrt{n}} \leq \frac{1}{p} \leq \frac{A_n}{1-4.5/\sqrt{n}}.$$

□

Examples:

Confidence Interval for $1/p$ in $G(p)$

Let X_n be i.i.d. $G(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

Theorem:

$$[\frac{A_n}{1+4.5/\sqrt{n}}, \frac{A_n}{1-4.5/\sqrt{n}}] \text{ is a 95\%-CI for } \frac{1}{p}.$$

Proof: We know that

$$Pr[\mu \in [A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}]] \geq 95\%.$$

Here, $\mu = \frac{1}{p}$ and $\sigma = \frac{\sqrt{1-p}}{p} \leq \frac{1}{p}$. Hence,

$$Pr[\frac{1}{p} \in [A_n - 4.5\frac{1}{p\sqrt{n}}, A_n + 4.5\frac{1}{p\sqrt{n}}]] \geq 95\%.$$

Now, $A_n - 4.5\frac{1}{p\sqrt{n}} \leq \frac{1}{p} \leq \frac{1}{p} \leq A_n + 4.5\frac{1}{p\sqrt{n}}$ is equivalent to

$$\frac{A_n}{1+4.5/\sqrt{n}} \leq \frac{1}{p} \leq \frac{A_n}{1-4.5/\sqrt{n}}.$$

$\square$

Examples: $[0.7A_{100}, 1.8A_{100}]$

Confidence Interval for $1/p$ in $G(p)$

Let X_n be i.i.d. $G(p)$. Define $A_n = (X_1 + \cdots + X_n)/n$.

Theorem:

$$[\frac{A_n}{1 + 4.5/\sqrt{n}}, \frac{A_n}{1 - 4.5/\sqrt{n}}] \text{ is a 95\%-CI for } \frac{1}{p}.$$

Proof: We know that

$$Pr[\mu \in [A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}]] \geq 95\%.$$

Here, $\mu = \frac{1}{p}$ and $\sigma = \frac{\sqrt{1-p}}{p} \leq \frac{1}{p}$. Hence,

$$Pr[\frac{1}{p} \in [A_n - 4.5\frac{1}{p\sqrt{n}}, A_n + 4.5\frac{1}{p\sqrt{n}}]] \geq 95\%.$$

Now, $A_n - 4.5\frac{1}{p\sqrt{n}} \leq \frac{1}{p} \leq \frac{1}{p} \leq A_n + 4.5\frac{1}{p\sqrt{n}}$ is equivalent to

$$\frac{A_n}{1 + 4.5/\sqrt{n}} \leq \frac{1}{p} \leq \frac{A_n}{1 - 4.5/\sqrt{n}}.$$

$\square$

Examples: $[0.7A_{100}, 1.8A_{100}]$ and $[0.96A_{10000}, 1.05A_{10000}]$.

Which Coin is Better?

Which Coin is Better?

You are given coin A and coin B.

Which Coin is Better?

You are given coin A and coin B. You want to find out which one has a larger $Pr[H]$.

Which Coin is Better?

You are given coin A and coin B. You want to find out which one has a larger $Pr[H]$. Let p_A and p_B be the values of $Pr[H]$ for the two coins.

Which Coin is Better?

You are given coin A and coin B. You want to find out which one has a larger $Pr[H]$. Let p_A and p_B be the values of $Pr[H]$ for the two coins.

Approach:

Which Coin is Better?

You are given coin A and coin B. You want to find out which one has a larger $Pr[H]$. Let p_A and p_B be the values of $Pr[H]$ for the two coins.

Approach:

- Flip each coin n times.

Which Coin is Better?

You are given coin A and coin B. You want to find out which one has a larger $Pr[H]$. Let p_A and p_B be the values of $Pr[H]$ for the two coins.

Approach:

- Flip each coin n times.
- Let A_n be the fraction of Hs for coin A and B_n for coin B.

Which Coin is Better?

You are given coin A and coin B. You want to find out which one has a larger $Pr[H]$. Let p_A and p_B be the values of $Pr[H]$ for the two coins.

Approach:

- Flip each coin n times.
- Let A_n be the fraction of Hs for coin A and B_n for coin B.
- Assume $A_n > B_n$. It is tempting to think that $p_A > p_B$.

Which Coin is Better?

You are given coin A and coin B. You want to find out which one has a larger $Pr[H]$. Let p_A and p_B be the values of $Pr[H]$ for the two coins.

Approach:

- Flip each coin n times.
- Let A_n be the fraction of Hs for coin A and B_n for coin B.
- Assume $A_n > B_n$. It is tempting to think that $p_A > p_B$. Confidence?

Analysis:

Which Coin is Better?

You are given coin A and coin B. You want to find out which one has a larger $Pr[H]$. Let p_A and p_B be the values of $Pr[H]$ for the two coins.

Approach:

- Flip each coin n times.
- Let A_n be the fraction of Hs for coin A and B_n for coin B.
- Assume $A_n > B_n$. It is tempting to think that $p_A > p_B$. Confidence?

Analysis: Note that

$E[A_n - B_n] =$

Which Coin is Better?

You are given coin A and coin B. You want to find out which one has a larger $Pr[H]$. Let p_A and p_B be the values of $Pr[H]$ for the two coins.

Approach:

- Flip each coin n times.
- Let A_n be the fraction of Hs for coin A and B_n for coin B.
- Assume $A_n > B_n$. It is tempting to think that $p_A > p_B$. Confidence?

Analysis: Note that

$E[A_n - B_n] = p_A - p_B$ and $var(A_n - B_n) =$

Which Coin is Better?

You are given coin A and coin B. You want to find out which one has a larger $Pr[H]$. Let p_A and p_B be the values of $Pr[H]$ for the two coins.

Approach:

- Flip each coin n times.
- Let A_n be the fraction of Hs for coin A and B_n for coin B.
- Assume $A_n > B_n$. It is tempting to think that $p_A > p_B$. Confidence?

Analysis: Note that

$$E[A_n - B_n] = p_A - p_B \text{ and } var(A_n - B_n) = \frac{1}{n}(p_A(1-p_A) + p_B(1-p_B)) \leq \frac{1}{2n}.$$

Which Coin is Better?

You are given coin A and coin B. You want to find out which one has a larger $Pr[H]$. Let p_A and p_B be the values of $Pr[H]$ for the two coins.

Approach:

- Flip each coin n times.
- Let A_n be the fraction of Hs for coin A and B_n for coin B.
- Assume $A_n > B_n$. It is tempting to think that $p_A > p_B$. Confidence?

Analysis: Note that

$$E[A_n - B_n] = p_A - p_B \text{ and } var(A_n - B_n) = \frac{1}{n}(p_A(1-p_A) + p_B(1-p_B)) \leq \frac{1}{2n}.$$

Thus, $Pr[|A_n - B_n - (p_A - p_B)| > \varepsilon] \leq \frac{1}{2n\varepsilon^2}$,

Which Coin is Better?

You are given coin A and coin B. You want to find out which one has a larger $Pr[H]$. Let p_A and p_B be the values of $Pr[H]$ for the two coins.

Approach:

- Flip each coin n times.
- Let A_n be the fraction of Hs for coin A and B_n for coin B.
- Assume $A_n > B_n$. It is tempting to think that $p_A > p_B$. Confidence?

Analysis: Note that

$$E[A_n - B_n] = p_A - p_B \text{ and } var(A_n - B_n) = \frac{1}{n}(p_A(1-p_A) + p_B(1-p_B)) \leq \frac{1}{2n}.$$

Thus, $Pr[|A_n - B_n - (p_A - p_B)| > \varepsilon] \leq \frac{1}{2n\varepsilon^2}$,

$$Pr[p_A - p_B \in [A_n - B_n - \varepsilon, A_n - B_n + \varepsilon]] \geq 1 - \frac{1}{2n\varepsilon^2}$$

,

Which Coin is Better?

You are given coin A and coin B. You want to find out which one has a larger $Pr[H]$. Let p_A and p_B be the values of $Pr[H]$ for the two coins.

Approach:

- Flip each coin n times.
- Let A_n be the fraction of Hs for coin A and B_n for coin B.
- Assume $A_n > B_n$. It is tempting to think that $p_A > p_B$. Confidence?

Analysis: Note that

$$E[A_n - B_n] = p_A - p_B \text{ and } var(A_n - B_n) = \frac{1}{n}(p_A(1-p_A) + p_B(1-p_B)) \leq \frac{1}{2n}.$$

Thus, $Pr[|A_n - B_n - (p_A - p_B)| > \varepsilon] \leq \frac{1}{2n\varepsilon^2}$,

$$Pr[p_A - p_B \in [A_n - B_n - \varepsilon, A_n - B_n + \varepsilon]] \geq 1 - \frac{1}{2n\varepsilon^2}$$

,

Which Coin is Better?

You are given coin A and coin B. You want to find out which one has a larger $Pr[H]$. Let p_A and p_B be the values of $Pr[H]$ for the two coins.

Approach:

- Flip each coin n times.
- Let A_n be the fraction of Hs for coin A and B_n for coin B.
- Assume $A_n > B_n$. It is tempting to think that $p_A > p_B$. Confidence?

Analysis: Note that

$$E[A_n - B_n] = p_A - p_B \text{ and } var(A_n - B_n) = \frac{1}{n}(p_A(1-p_A) + p_B(1-p_B)) \leq \frac{1}{2n}.$$

Thus, $Pr[|A_n - B_n - (p_A - p_B)| > \varepsilon] \leq \frac{1}{2n\varepsilon^2}$,

$$Pr[p_A - p_B \in [A_n - B_n - \varepsilon, A_n - B_n + \varepsilon]] \geq 1 - \frac{1}{2n\varepsilon^2}$$

,

or

Which Coin is Better?

You are given coin A and coin B. You want to find out which one has a larger $Pr[H]$. Let p_A and p_B be the values of $Pr[H]$ for the two coins.

Approach:

- Flip each coin n times.
- Let A_n be the fraction of Hs for coin A and B_n for coin B.
- Assume $A_n > B_n$. It is tempting to think that $p_A > p_B$. Confidence?

Analysis: Note that

$$E[A_n - B_n] = p_A - p_B \text{ and } var(A_n - B_n) = \frac{1}{n}(p_A(1-p_A) + p_B(1-p_B)) \leq \frac{1}{2n}.$$

Thus, $Pr[|A_n - B_n - (p_A - p_B)| > \varepsilon] \leq \frac{1}{2n\varepsilon^2}$,

$$Pr[p_A - p_B \in [A_n - B_n - \varepsilon, A_n - B_n + \varepsilon]] \geq 1 - \frac{1}{2n\varepsilon^2}$$

,

or

"p-value" of "$p_A \geq p_B$" $\leq \frac{1}{2n(A_n - B_n)^2}$.

Which Coin is Better?

You are given coin A and coin B. You want to find out which one has a larger $Pr[H]$. Let p_A and p_B be the values of $Pr[H]$ for the two coins.

Approach:

- Flip each coin n times.
- Let A_n be the fraction of Hs for coin A and B_n for coin B.
- Assume $A_n > B_n$. It is tempting to think that $p_A > p_B$. Confidence?

Analysis: Note that

$$E[A_n - B_n] = p_A - p_B \text{ and } var(A_n - B_n) = \frac{1}{n}(p_A(1-p_A) + p_B(1-p_B)) \leq \frac{1}{2n}.$$

Thus, $Pr[|A_n - B_n - (p_A - p_B)| > \varepsilon] \leq \frac{1}{2n\varepsilon^2}$,

$$Pr[p_A - p_B \in [A_n - B_n - \varepsilon, A_n - B_n + \varepsilon]] \geq 1 - \frac{1}{2n\varepsilon^2}$$

,

or

"p-value" of "$p_A \geq p_B$" $\leq \frac{1}{2n(A_n - B_n)^2}$.

Unknown σ

Unknown σ

For $B(p)$, we wanted to estimate p.

Unknown σ

For $B(p)$, we wanted to estimate p. The CI requires $\sigma = \sqrt{p(1-p)}$.

Unknown σ

For $B(p)$, we wanted to estimate p. The CI requires $\sigma = \sqrt{p(1-p)}$. We replaced σ by an upper bound: $1/2$.

Unknown σ

For $B(p)$, we wanted to estimate p. The CI requires $\sigma = \sqrt{p(1-p)}$. We replaced σ by an upper bound: $1/2$.

In some applications, it may be OK to replace σ^2 by the following sample variance:

Unknown σ

For $B(p)$, we wanted to estimate p. The CI requires $\sigma = \sqrt{p(1-p)}$. We replaced σ by an upper bound: $1/2$.

In some applications, it may be OK to replace σ^2 by the following sample variance:

$$s_n^2 := \frac{1}{n}\sum_{m=1}^{n}(X_m - A_n)^2.$$

Unknown σ

For $B(p)$, we wanted to estimate p. The CI requires $\sigma = \sqrt{p(1-p)}$. We replaced σ by an upper bound: $1/2$.

In some applications, it may be OK to replace σ^2 by the following sample variance:

$$s_n^2 := \frac{1}{n} \sum_{m=1}^{n} (X_m - A_n)^2.$$

However, in some cases, this is dangerous!

Unknown σ

For $B(p)$, we wanted to estimate p. The CI requires $\sigma = \sqrt{p(1-p)}$. We replaced σ by an upper bound: $1/2$.

In some applications, it may be OK to replace σ^2 by the following sample variance:

$$s_n^2 := \frac{1}{n} \sum_{m=1}^{n} (X_m - A_n)^2.$$

However, in some cases, this is dangerous! The theory says it is OK if the distribution of X_n is nice (Gaussian).

Unknown σ

For $B(p)$, we wanted to estimate p. The CI requires $\sigma = \sqrt{p(1-p)}$. We replaced σ by an upper bound: $1/2$.

In some applications, it may be OK to replace σ^2 by the following sample variance:

$$s_n^2 := \frac{1}{n}\sum_{m=1}^{n}(X_m - A_n)^2.$$

However, in some cases, this is dangerous! The theory says it is OK if the distribution of X_n is nice (Gaussian). This is used regularly in practice.

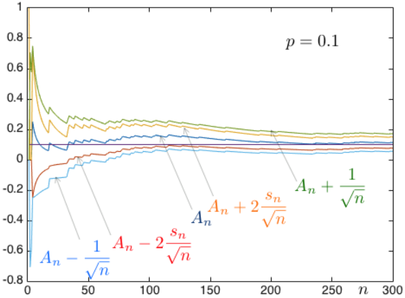
Unknown σ

For $B(p)$, we wanted to estimate p. The CI requires $\sigma = \sqrt{p(1-p)}$. We replaced σ by an upper bound: $1/2$.

In some applications, it may be OK to replace σ^2 by the following sample variance:

$$s_n^2 := \frac{1}{n}\sum_{m=1}^{n}(X_m - A_n)^2.$$

However, in some cases, this is dangerous! The theory says it is OK if the distribution of X_n is nice (Gaussian). This is used regularly in practice. However, be aware of the risk.



Summary

$\boxed{\text{Confidence Intervals}}$

Summary

$\boxed{\text{Confidence Intervals}}$

1. Estimates without confidence level are useless!

Summary

$\boxed{\text{Confidence Intervals}}$

1. Estimates without confidence level are useless!
2. $[a, b]$ is a 95%-CI for θ if

Summary

$\boxed{\text{Confidence Intervals}}$

1. Estimates without confidence level are useless!
2. $[a, b]$ is a 95%-CI for θ if $Pr[\theta \in [a, b]] \geq 95\%$.

Summary

$\boxed{\text{Confidence Intervals}}$

1. Estimates without confidence level are useless!
2. $[a,b]$ is a 95%-CI for θ if $Pr[\theta \in [a,b]] \geq 95\%$.
3. Using Chebyshev: $[A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}]$ is a 95%-CI for μ.

Summary

$\boxed{\text{Confidence Intervals}}$

1. Estimates without confidence level are useless!
2. $[a,b]$ is a 95%-CI for θ if $Pr[\theta \in [a,b]] \geq 95\%$.
3. Using Chebyshev: $[A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}]$ is a 95%-CI for μ.
4. Using CLT, we will replace 4.5 by 2.

Summary

$\boxed{\text{Confidence Intervals}}$

1. Estimates without confidence level are useless!
2. $[a,b]$ is a 95%-CI for θ if $Pr[\theta \in [a,b]] \geq 95\%$.
3. Using Chebyshev: $[A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}]$ is a 95%-CI for μ.
4. Using CLT, we will replace 4.5 by 2.
5. When σ is not known, one can replace it by an upper bound.

Summary

Confidence Intervals

1. Estimates without confidence level are useless!
2. $[a,b]$ is a 95%-CI for θ if $Pr[\theta \in [a,b]] \geq 95\%$.
3. Using Chebyshev: $[A_n - 4.5\sigma/\sqrt{n}, A_n + 4.5\sigma/\sqrt{n}]$ is a 95%-CI for μ.
4. Using CLT, we will replace 4.5 by 2.
5. When σ is not known, one can replace it by an upper bound.
6. Examples: $B(p), G(p)$, which coin is better?
7. In some cases, one can replace σ by the empirical standard deviation.