CS70

Coupon Collecting: Fun with harmonic numbers!

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Variance/ Covariance.

Time to collect coupons

X-time to get n coupons.

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Pr[“get second coupon”|“got milk ”]

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Coupons: Poll

Collect n coupons!

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What's True?

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What's True?

(A) $X_1 = \frac{n}{n} = 1$.
(B) $X_2 = \frac{n}{n-1}$.
(C) $Pr[\text{getting second}|\text{got first}] = \frac{n-1}{n}$.
(D) $E[X_2] = \frac{n}{n-1}$.
(E) $E[X_n] = n$.
(F) $\sum_i E[X_i] = \sum_{i=0}^{n-1} \frac{n}{n-i}$
(G) $\sum_i E[X_i] = \sum_{i=1}^{n} \frac{1}{n}$

Review: Harmonic sum

$$H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \int_1^n \frac{1}{x} dx = \ln(n).$$

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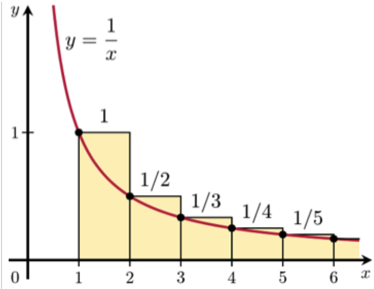

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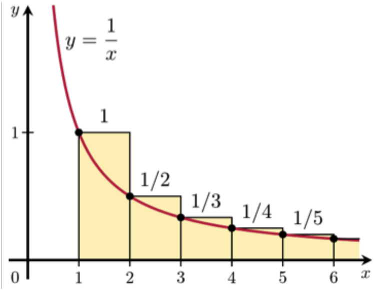

A good approximation is

$H(n) \approx \ln(n) + \gamma$ where $\gamma \approx 0.58$ (Euler-Mascheroni constant).

Harmonic sum: Paradox

Consider this stack of cards (no glue!):

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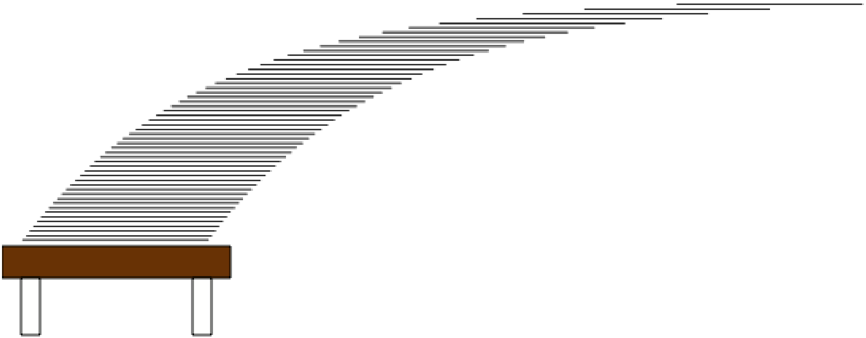

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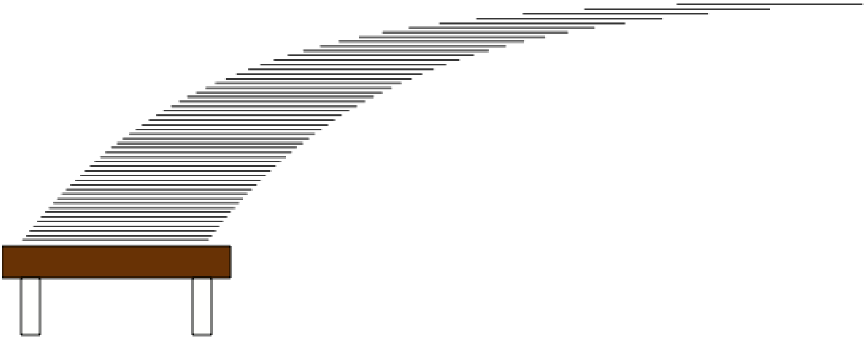

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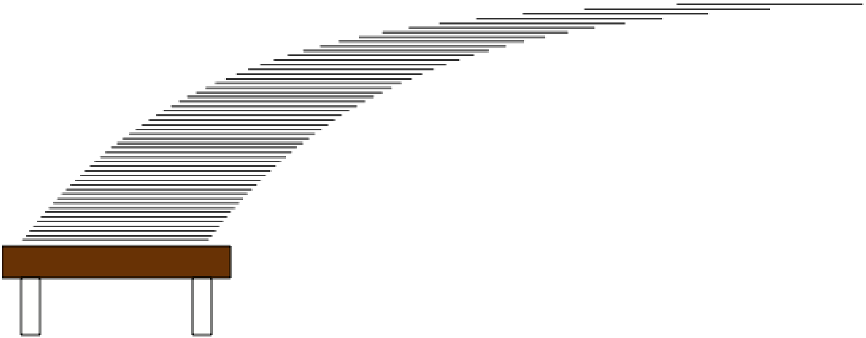


If each card has length 2, the stack can extend $H(n)$ to the right of the table. As n increases, you can go as far as you want!

Paradox

par·a·dox

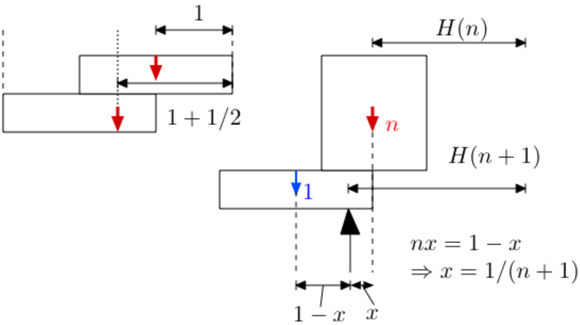
/ˈperəˌdäks/

noun

a statement or proposition that, despite sound (or apparently sound) reasoning from acceptable premises, leads to a conclusion that seems senseless, logically unacceptable, or self-contradictory.
"a potentially serious conflict between quantum mechanics and the general theory of relativity known as the information paradox"

- a seemingly absurd or self-contradictory statement or proposition that when investigated or explained may prove to be well founded or true.
  "in a paradox, he has discovered that stepping back from his job has increased the rewards he gleans from it"
  synonyms: contradiction, contradiction in terms, self-contradiction, inconsistency, incongruity; More
- a situation, person, or thing that combines contradictory features or qualities.
  "the mingling of deciduous trees with elements of desert flora forms a fascinating ecological paradox"

Stacking



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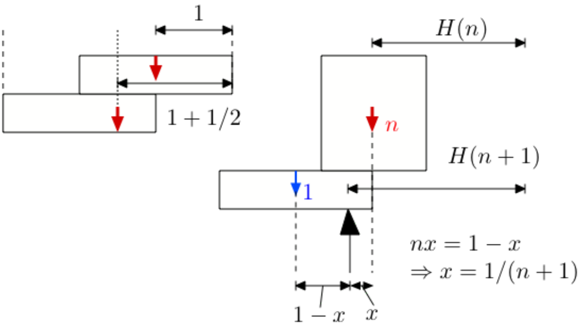

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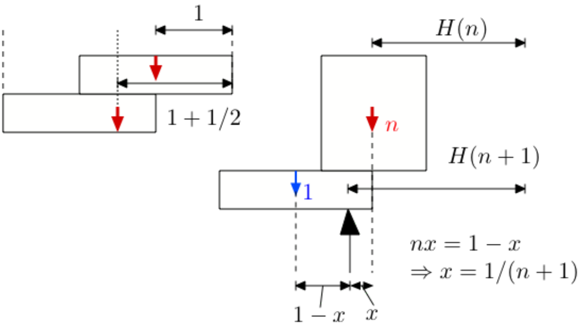

The cards have width 2. Induction shows that the center of gravity after *n* cards is *H*(*n*) away from the right-most edge.
Video.

Calculating $E[g(X)]$: LOTUS

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&= \sum_x g(x) Pr[X = x].
\end{aligned}$$

$\square$

Poll.

Which is LOTUS?

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(A) $E[X] = \sum_{x \in \text{Range}(X)} g(x) Pr[g(X) = g(x)]$
(B) $E[X] = \sum_{x \in \text{Range}(X)} g(x) Pr[X = x]$
(C) $E[X] = \sum_{x \in \text{Range}(g)} x Pr[g(X) = x]$

Geometric Distribution.

Experiment: flip a coin with heads prob. p. until Heads.
Random Variable X: number of flips.

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(A) Distribution of $X \sim G(p)$: $Pr[X = i] = (1-p)^{i-1}p$.

Geometric Distribution: Memoryless - Interpretation

$$Pr[X > n+m | X > n] = Pr[X > m], m, n \geq 0.$$

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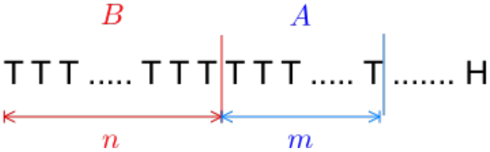

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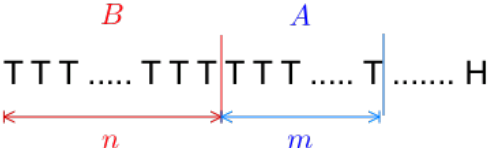

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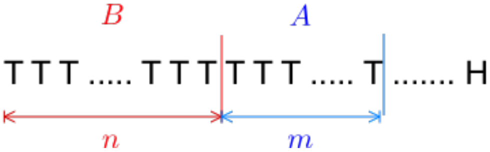

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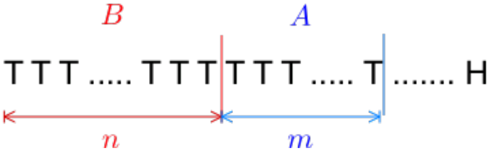

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The coin is memoryless , therefore, so is X.
Independent coin: $Pr[H|'anyprevioussetofcointosses'] = p$

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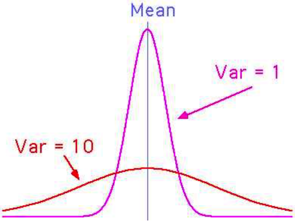

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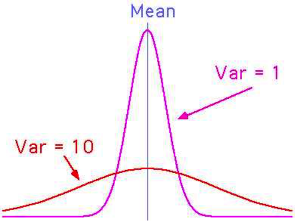

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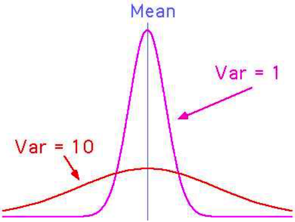

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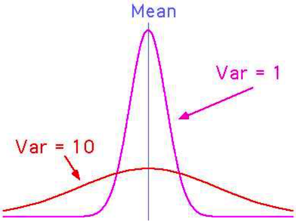

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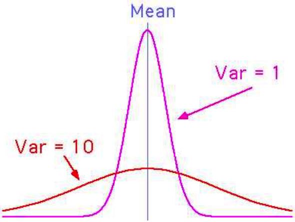

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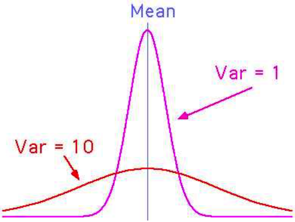

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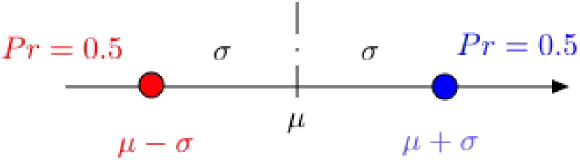

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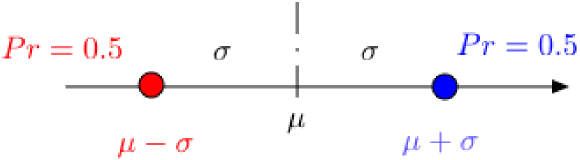

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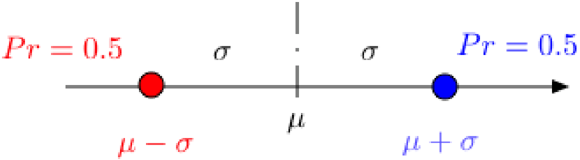

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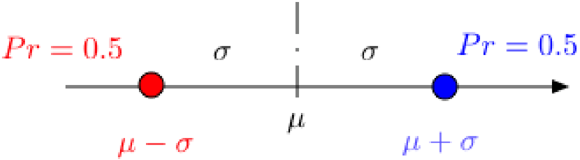

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Exercise: How big can you make $\frac{\sigma(X)}{E[|X - E[X]|]}$?

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This gives

$$var(X) = \frac{1 + 3n + 2n^2}{6} - \frac{(n+1)^2}{4} = \frac{n^2 - 1}{12}.$$

(Sort of $\int_0^{1/2} x^2 dx = \frac{x^3}{3}$.)

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(A) X_i and X_j are independent.
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(C) $Pr[X_iX_j] = \frac{(n-2)!}{n!}$
(D) $X_i^2 = X_i$.

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&= (\sum_a aPr[X = a])(\sum_b bPr[Y = b]) \\
&= E[X]E[Y]
\end{aligned}$$

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Theorem:
If X and Y are independent, then

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$\square$

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Flip coin with heads probability p.

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$$\begin{aligned}E[(X - E[X])(Y - E[Y])] &= E[XY - E[X]Y - XE[Y] + E[X]E[Y]] \\ &= E[XY] - E[X]E[Y] - E[X]E[Y] + E[X]E[Y] \\ &= E[XY] - E[X]E[Y].\end{aligned}$$

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Definition The correlation of X, Y, $Cor(X,Y)$ is

$$corr(X,Y) : \frac{cov(X,Y)}{\sigma(X)\sigma(Y)}.$$

Theorem: $-1 \leq corr(X,Y) \leq 1$.
Proof: Idea: $(a-b)^2 > 0 \rightarrow a^2 + b^2 \geq 2ab$.

Simple case: $E[X] = E[Y] = 0$ and $E[X^2] = E[Y^2] = 1$.

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$E[(X-Y)^2] = E[X^2] + E[Y^2] - 2E[XY] = 2(1 - E[XY]) \geq 0$
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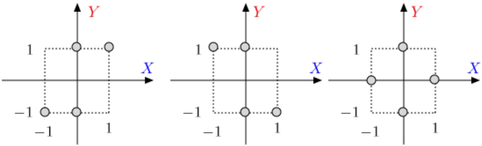
$\square$

Shifting and scaling doesn't change correlation.

Examples of Covariance

Four equally likely pairs of values



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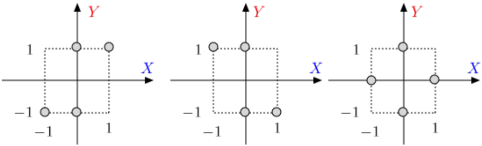

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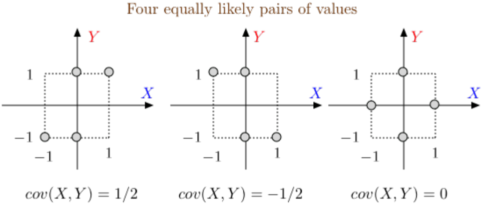

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When $cov(X, Y) > 0$, the RVs X and Y tend to be large or small together.

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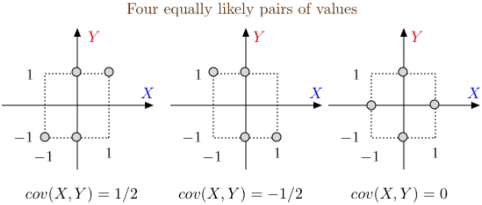

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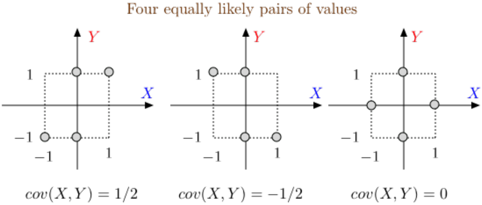

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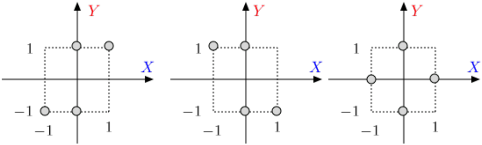

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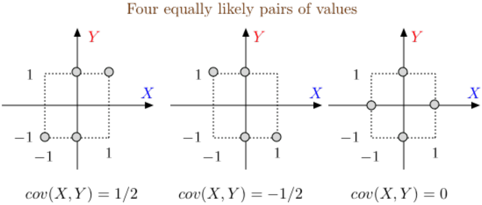

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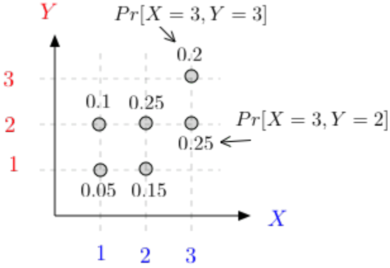

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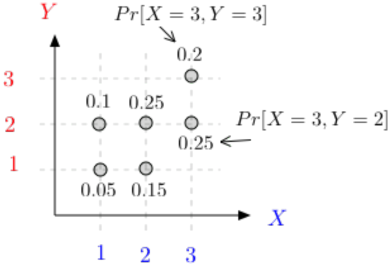

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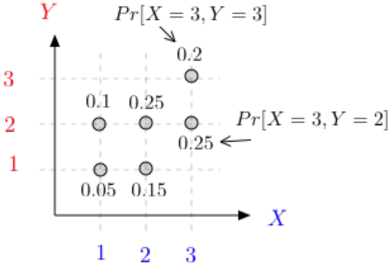

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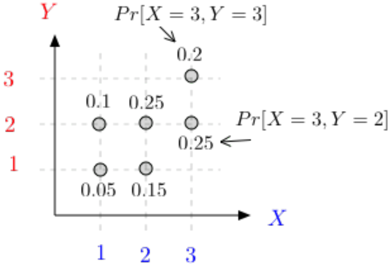

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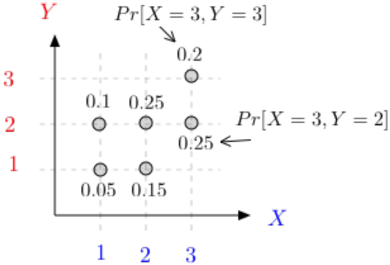

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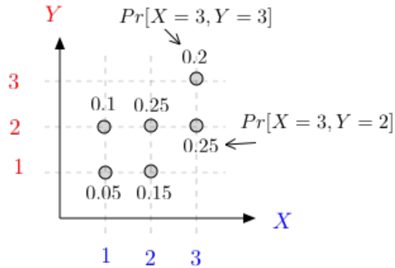

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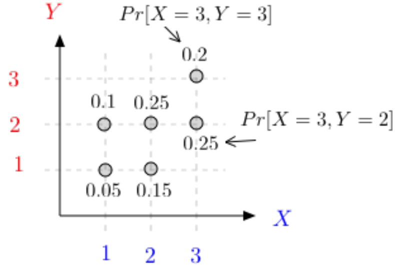

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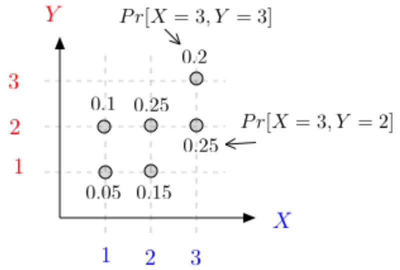

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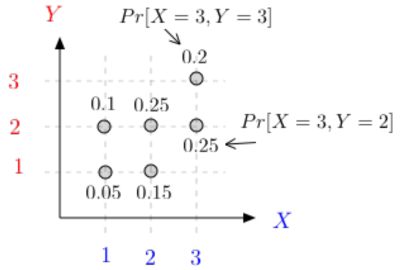

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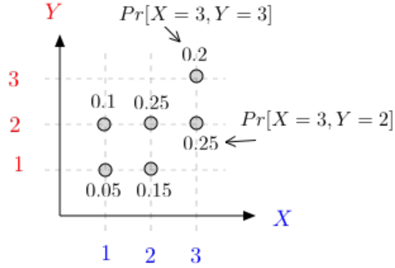

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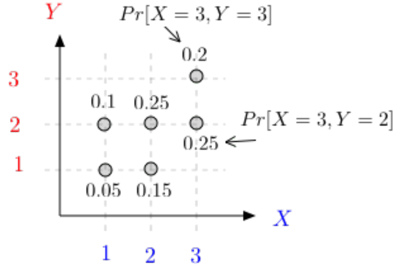

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Random Variables so far.

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Uniform: $X \sim U\{1, \ldots, n\}$ $E[X] = \frac{n+1}{2}$, $Var(X) = \frac{n^2-1}{12}$.

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