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Stars and Bars Poll

Mark whats correct.

(A) ways to split n dollars among k: $\binom{n+k-1}{k-1}$

(B) ways to split k dollars among n: $\binom{k+n-1}{n-1}$

(C) ways to split 5 dollars among 3: $\binom{7}{5}$

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All correct.

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5 indistinguishable balls into 52 bins only one ball in each bin
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Example: Poker hands.

5 indistinguishable balls into 3 bins
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5 indistinguishable balls into 3 bins
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Dividing 5 dollars among Alice, Bob and Eve.

Poll

Mark whats correct.

k Balls in n bins.

dis == distinguishiable unique = one ball in each bin.

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Pascal’s Triangle

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	0	
1		1

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$$\begin{array}{ccccc} & & 0 & & \\ & 1 & & 1 & \\ 1 & & 2 & & 1 \end{array}$$

Pascal's Triangle

			0			
		1		1		
	1		2		1	
1		3		3		1

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				0				
			1		1			
		1		2		1		
	1		3		3		1	
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Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.

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Simplify: collect all terms corresponding to x^k.
  Coefficient of x^k $\binom{n}{k}$: choose k terms with x in product.

Pascal's Triangle

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.

Foil (4 terms) on steroids:
 2^n terms: choose 1 or x from each term $(1+x)$.

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Pascal's rule $\implies \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$?

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How many contain the first element?

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Chose first element, need $k-1$ more from remaining n elements.

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Sum Rule: size of union of disjoint sets of objects.

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So, $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$. $\square$

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Must choose $k-1$ elements from $n-i$ remaining elements.

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Binomial Theorem: $x = 1$

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Simple Inclusion/Exclusion

Sum Rule: For disjoint sets S and T, $|S \cup T| = |S| + |T|$
Used to reason about all subsets
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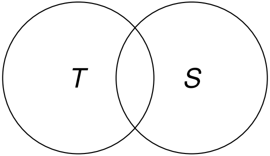

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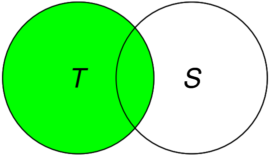

In T. $\implies |T|$

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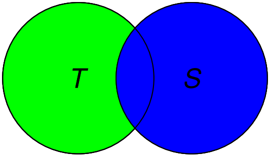

In T. $\implies |T|$
In S. $\implies + \ |S|$

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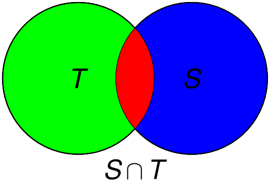

In T. $\implies$ $|T|$
In S. $\implies$ $+$ $|S|$
Elements in $S \cap T$ are counted twice.

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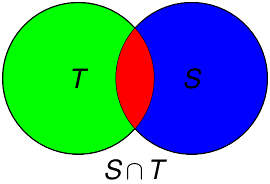

In T. $\implies |T|$
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Subtract. $\implies -|S \cap T|$

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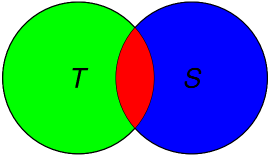

$S \cap T$

In T. $\Longrightarrow$ $|T|$

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Example: How many 10-digit phone numbers have 7 as their first or second digit?

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$S \cap T$ = phone numbers with 7 as first and second digit.

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$S \cap T$ = phone numbers with 7 as first and second digit. $|S \cap T| = 10^8$.

Simple Inclusion/Exclusion

Sum Rule: For disjoint sets S and T, $|S \cup T| = |S| + |T|$
Used to reason about all subsets
by adding number of subsets of size 1, 2, 3,...

Also reasoned about subsets that contained
or didn't contain an element. (E.g., first element, first i elements.)

Inclusion/Exclusion Rule: For any S and T,
$|S \cup T| = |S| + |T| - |S \cap T|$.

Example: How many 10-digit phone numbers have 7 as their first or second digit?

S = phone numbers with 7 as first digit.$|S| = 10^9$

T = phone numbers with 7 as second digit. $|T| = 10^9$.

$S \cap T$ = phone numbers with 7 as first and second digit. $|S \cap T| = 10^8$.

Answer: $|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$.

Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
$\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|$.

Inclusion/Exclusion

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Idea: For $n = 3$ how many times is an element counted?

Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
$\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|$.

Idea: For $n = 3$ how many times is an element counted?

Consider $x \in A_i \cap A_j$.

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$|A_1 \cup \cdots \cup A_n| =$
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Consider $x \in A_i \cap A_j$.

x counted once for $|A_i|$ and once for $|A_j|$.

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x counted once for $|A_i|$ and once for $|A_j|$.

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Total: 2 -1 = 1.

Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
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Consider $x \in A_1 \cap A_2 \cap A_3$

Inclusion/Exclusion

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Total: 2 -1 = 1.

Consider $x \in A_1 \cap A_2 \cap A_3$

x counted once in each term: $|A_1|, |A_2|, |A_3|$.

Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
$\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|$.

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Total: 3 - 3 + 1 = 1.

Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
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Total: 3 - 3 + 1 = 1.

Formulaically: x is in intersection of three sets.

Inclusion/Exclusion

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x counted once in each term: $|A_1|, |A_2|, |A_3|$.

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x added once in $|A_1 \cap A_2 \cap A_3|$.

Total: 3 - 3 + 1 = 1.

Formulaically: x is in intersection of three sets.

3 for terms of form $|A_i|$, $\binom{3}{2}$ for terms of form $|A_i \cap A_j|$.

Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
$\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|$.

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Consider $x \in A_1 \cap A_2 \cap A_3$

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Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
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3 for terms of form $|A_i|$, $\binom{3}{2}$ for terms of form $|A_i \cap A_j|$.

$\binom{3}{3}$ for terms of form $|A_i \cap A_j \cap A_k|$.

Total: $\binom{3}{1} - \binom{3}{2} + \binom{3}{3} = 1$.

Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
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Idea: how many times is each element counted?
Element x in m sets: $x \in A_{i_1} \cap A_{i_2} \cdots \cap A_{i_m}$.

Inclusion/Exclusion

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Binomial Theorem:
$(x+y)^m = \binom{m}{0} x^m + \binom{m}{1} x^{m-1} y + \binom{m}{2} x^{m-2} y^2 + \cdots \binom{m}{m} y^m$.

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  Proof: m factors in product: $(x+y)(x+y)\cdots(x+y)$.

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Get a term $x^{m-i}y^i$ by choosing i factors to use for y.

Inclusion/Exclusion

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are $\binom{m}{i}$ ways to choose factors where y is provided.

Inclusion/Exclusion

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For $x = 1, y = -1$,

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$0 = (1-1)^m = \binom{m}{0} - \binom{m}{1} + \binom{m}{2} \cdots + (-1)^m \binom{m}{m}$

$\implies \quad 1 = \binom{m}{0} = \binom{m}{1} - \binom{m}{2} \cdots + (-1)^{m-1} \binom{m}{m}$.

Inclusion/Exclusion

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$\implies \quad 1 = \binom{m}{0} = \binom{m}{1} - \binom{m}{2} \cdots + (-1)^{m-1}\binom{m}{m}$.

Each element counted once!

Summary.

First Rule of counting:

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Summary.

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n_i possibilitities for ith choice :

Summary.

First Rule of counting: Objects from a sequence of choices:
n_i possibilitities for ith choice : $n_1 \times n_2 \times \cdots \times n_k$ objects.

Summary.

First Rule of counting: Objects from a sequence of choices:

n_i possibilitities for ith choice : $n_1 \times n_2 \times \cdots \times n_k$ objects.

Second Rule of counting:

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Disjoint

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Disjoint – so add!

Poll: How big is infinity?

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Mark what's true.
(A) There are more real numbers than natural numbers.
(B) There are more rational numbers than natural numbers.
(C) There are more integers than natural numbers.
(D) pairs of natural numbers $>>$ natural numbers.

Same Size. Poll.

Two sets are the same size?

Same Size. Poll.

Two sets are the same size?

(A) Bijection between the sets.
(B) Count the objects and get the same number. same size.
(C) Counting to infinity is hard.

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(A), (B).

Same Size. Poll.

Two sets are the same size?

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(A), (B).
(C)?

Next up: how big is infinity.

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- Countable
- Countably infinite.
- Enumeration

How big are the reals or the integers?

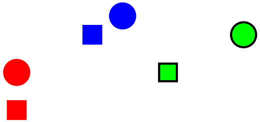
Infinite!

How big are the reals or the integers?

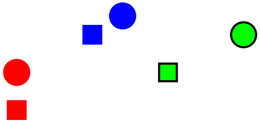
Infinite!

Is one bigger or smaller?

Same size?



Same size?



Same number?

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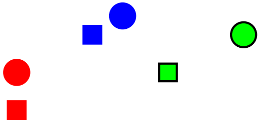
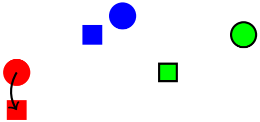


Same number?

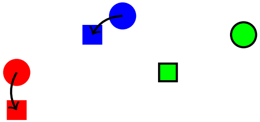
Make a function f : Circles $\to$ Squares.

Same size?



Same number?
Make a function f : Circles $\rightarrow$ Squares.
f(red circle) $=$ red square

Same size?



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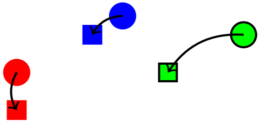
Same size?



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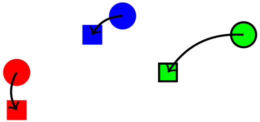

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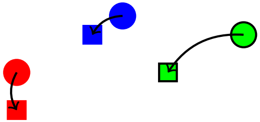
One to one.

Same size?



Same number?

Make a function f : Circles $\rightarrow$ Squares.

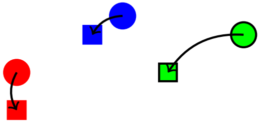
f(red circle) $=$ red square

f(blue circle) $=$ blue square

f(circle with black border) $=$ square with black border

One to one. Each circle mapped to different square.

Same size?



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One to One: For all $x, y \in D$, $x \neq y \implies f(x) \neq f(y)$.

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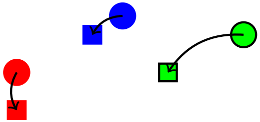

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Onto.

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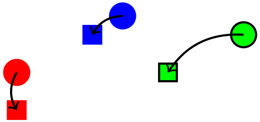

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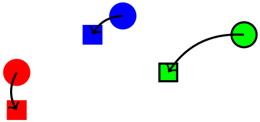

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Onto: For all $s \in R$, $\exists c \in D, s = f(c)$.

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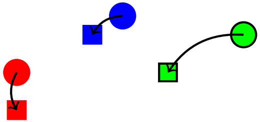

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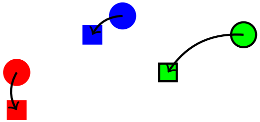

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Isomorphism principle: If there is $f : D \rightarrow R$ that is one to one and onto, then, $|D| = |R|$.

Isomorphism principle.

Given a function, $f : D \rightarrow R$.

Isomorphism principle.

Given a function, $f : D \rightarrow R$.

One to One:

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Given a function, $f : D \to R$.

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$f(\cdot)$ is a **bijection** if it is one to one and onto.

Isomorphism principle:
If there is a bijection $f : D \to R$ then $|D| = |R|$.

Countable.

How to count?

Countable.

How to count?

0,

Countable.

How to count?

0, 1,

Countable.

How to count?

0, 1, 2,

Countable.

How to count?

0, 1, 2, 3,

Countable.

How to count?

$0, 1, 2, 3, \ldots$

Countable.

How to count?

0, 1, 2, 3, $\ldots$

The Counting numbers.

Countable.

How to count?

0, 1, 2, 3, ...

The Counting numbers.
The natural numbers! N

Countable.

How to count?

0, 1, 2, 3, ...

The Counting numbers.
The natural numbers! N

Definition: S is **countable** if there is a bijection between S and some subset of N.

Countable.

How to count?

0, 1, 2, 3, ...

The Counting numbers.
The natural numbers! N

Definition: S is **countable** if there is a bijection between S and some subset of N.

If the subset of N is finite, S has finite **cardinality**.

Countable.

How to count?

0, 1, 2, 3, ...

The Counting numbers.
The natural numbers! N

Definition: S is **countable** if there is a bijection between S and some subset of N.

If the subset of N is finite, S has finite **cardinality**.

If the subset of N is infinite, S is **countably infinite**.

Where's 0?

Which is bigger?

Where's 0?

Which is bigger?
The positive integers, $\mathbb{Z}^+$, or the natural numbers, $\mathbb{N}$.

Where's 0?

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Natural numbers. 0,

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Natural numbers. $0, 1,$

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Positive integers. $1,$

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Where's 0?

More natural numbers!

Where's 0?

Which is bigger?
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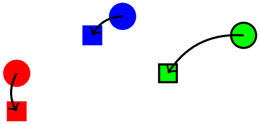

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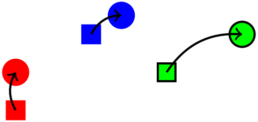

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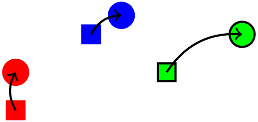

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Can prove equivalence either way.

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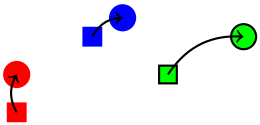

Inverse function!

Can prove equivalence either way.
Bijection to or from natural numbers implies countably infinite.

More large sets.

E - Even natural numbers?

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Evens are same size as all natural numbers.

All integers?

What about Integers, Z?

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$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

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Integers and naturals have same size!

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3	-2

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2	1
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4	2

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If infinite: bijection with N.

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"Output element of S",
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...
Any element x of S has *specific, finite* position in list.

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61A

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61A $\equiv$ streams!   Not Sp20/Fa20.

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All countably infinite sets have the same cardinality.

Enumeration example.

All binary strings.

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$B = \{0, 1\}^*$.

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$B = \{\phi; , 0, 00, 000, 0000, ...\}$
Never get to 1.

More fractions?

Enumerate the rational numbers in order...

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Can't list in "order".

Pairs of natural numbers.

Consider pairs of natural numbers: $N \times N$

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Enumerate in list:
$(0,0)$,

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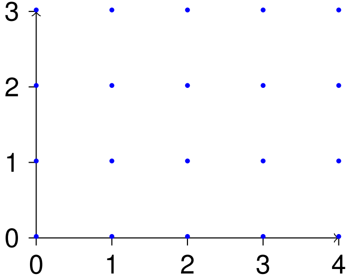

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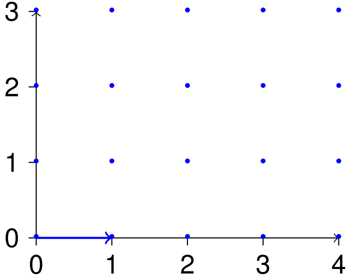

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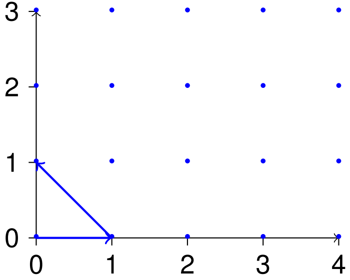

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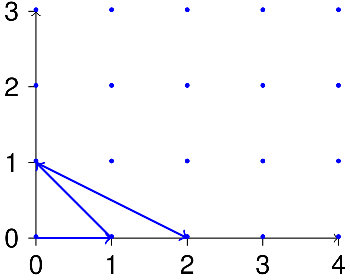

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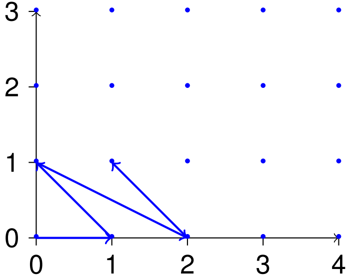

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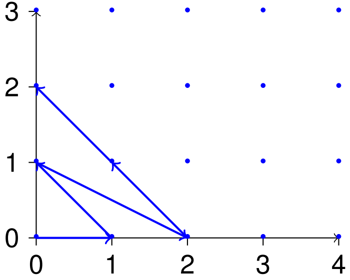

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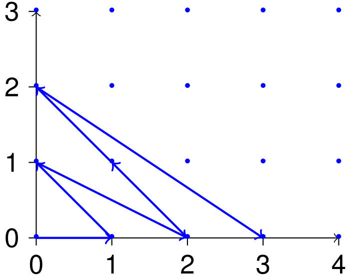

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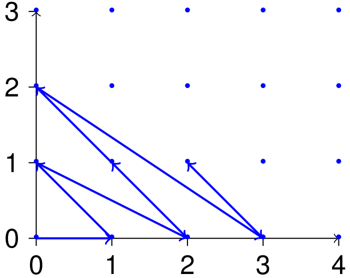

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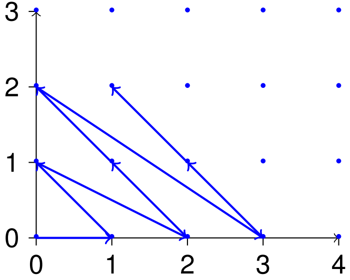

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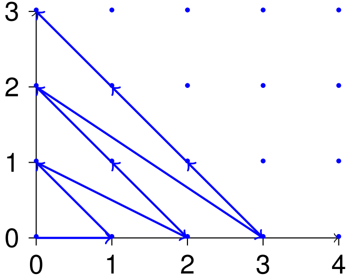

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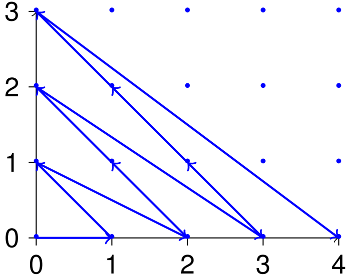

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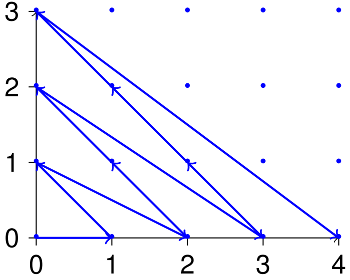

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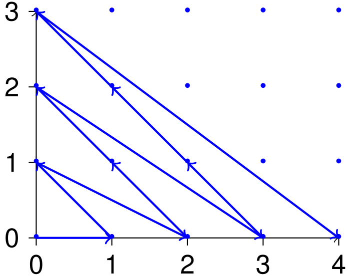

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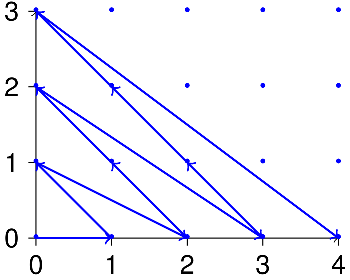

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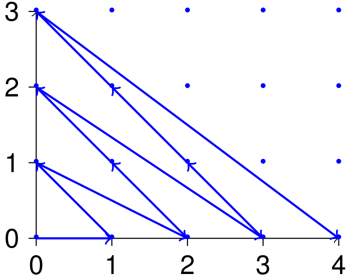

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Same size as the natural numbers!!

Poll.

Enumeration to get bijection with naturals?

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(A) Integers: First all negatives, then positives.
(B) Integers: By absolute value, break ties however.
(C) Pairs of naturals: by sum of values, break ties however.
(D) Pairs of naturals: by value of first element.
(E) Pairs of integers: by sum of values, break ties.
(F) Pairs of integers: by sum of absolute values, break ties.

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Enumeration to get bijection with naturals?

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(B),(C), (F).

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Positive rational number.

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Negative rationals are countable. (Same size as positive rationals.)

Put all rational numbers in a list.

First negative, then nonegative ??? No!

Repeatedly and alternatively take one from each list.

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Positive rational number.
Lowest terms: a/b
$a,b \in N$
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The rationals are countably infinite.

Real numbers..

Real numbers are same size as integers?

The reals.

Are the set of reals countable?

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⋮

Construct “diagonal” number: .77

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Subset $[0,1]$ is not countable!!

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(B) Integers are countable cuz, interleaving bijection.
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(B), (C)?, (D)