

Counting basics.

First rule: $n_1 \times n_2 \cdots \times n_3$.

k Samples with replacement from n items: n^k .

Sample without replacement: $\frac{n!}{(n-k)!}$

Second rule: when order doesn't matter divide..when possible.

Sample without replacement and order doesn't matter: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.
"n choose k"

One-to-one rule: equal in number if one-to-one correspondence.

Sample with replacement and order doesn't matter: $\binom{k+n-1}{n-1}$.

Balls in bins.

"k Balls in n bins" \equiv "k samples from n possibilities."

"indistinguishable balls" \equiv "order doesn't matter"

"only one ball in each bin" \equiv "without replacement"

5 balls into 10 bins

5 samples from 10 possibilities with replacement

Example: 5 digit numbers.

5 indistinguishable balls into 52 bins only one ball in each bin

5 samples from 52 possibilities without replacement

Example: Poker hands.

5 indistinguishable balls into 3 bins

5 samples from 3 possibilities with replacement and no order

Dividing 5 dollars among Alice, Bob and Eve.

Bijection: sums to 'k' \rightarrow stars and bars.

$$S = \{(n_1, n_2, n_3) : n_1 + n_2 + n_3 = 5\}$$

$$T = \{s \in \{ '|', '*' \} : |s| = 7, \text{ number of bars in } s = 2\}$$

$$f((n_1, n_2, n_3)) = *^{n_1} '| *^{n_2} '| *^{n_3}$$

Bijection:

argument: unique (n_1, n_2, n_3) from any s .

$$|S| = |T| = \binom{7}{2}.$$

Poll

Mark whats correct.

k Balls in n bins.

dis == distinguishable unique = one ball in each bin.

(A) dis $\Rightarrow n^k$

(B) dis, unique $\Rightarrow n!/(n-k)!$

(C) indis, unique $\Rightarrow \binom{n}{k}$

(D) dis, $\Rightarrow n!/(n-k)!$

(E) indis, $\Rightarrow \binom{n+k-1}{k-1}$

(F) dis, unique $\Rightarrow \binom{n}{k}$

Stars and Bars Poll

Mark whats correct.

(A) ways to split n dollars among k: $\binom{n+k-1}{k-1}$

(B) ways to split k dollars among n: $\binom{k+n-1}{n-1}$

(C) ways to split 5 dollars among 3: $\binom{7}{5}$

(D) ways to split 5 dollars among 3: $\binom{5+3-1}{3-1}$

All correct.

Sum Rule

Two indistinguishable jokers in 54 card deck.

How many 5 card poker hands?

Sum rule: Can sum over disjoint sets.

No jokers "exclusive" or One Joker "exclusive" or Two Jokers

$$\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.$$

Two distinguishable jokers in 54 card deck.

How many 5 card poker hands? Choose 4 cards plus one of 2 jokers!

$$\binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$$

Wait a minute! Same as choosing 5 cards from 54 or

$$\binom{54}{5}$$

Theorem: $\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$.

Algebraic Proof: Why? Just why? Especially on Tuesday!

Already have a **combinatorial proof**. \square

Combinatorial Proofs.

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

Proof: How many subsets of size k ? $\binom{n}{k}$

How many subsets of size k ?

Choose a subset of size $n-k$

and what's left out is a subset of size k .

Choosing a subset of size k is same

as choosing $n-k$ elements to not take.

$\implies \binom{n}{n-k}$ subsets of size k .

□

Pascal's Triangle

$$\begin{array}{ccccccc} & & & & 0 & & & & \\ & & & & 1 & & 1 & & \\ & & & & 1 & & 2 & & 1 \\ & & & & 1 & & 3 & & 3 & & 1 \\ & & & & 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Row n : coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.

Foil (4 terms) on steroids:

2^n terms: choose 1 or x from each term $(1+x)$.

Simplify: collect all terms corresponding to x^k .

Coefficient of x^k $\binom{n}{k}$: choose k terms with x in product.

$$\begin{array}{cccc} & & \binom{0}{0} & \\ & & \binom{1}{0} & \binom{1}{1} \\ & & \binom{2}{0} & \binom{2}{1} & \binom{2}{2} \\ \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} \end{array}$$

Pascal's rule $\implies \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Combinatorial Proof.

Theorem: $\binom{n}{k} = \binom{n-1}{k-1} + \cdots + \binom{n-1}{0}$.

Proof: Consider size k subset where i is the first element chosen.

$$\{1, \dots, i, \dots, n\}$$

Must choose $k-1$ elements from $n-i$ remaining elements.

$\implies \binom{n-i}{k-1}$ such subsets.

Add them up to get the total number of subsets of size k

which is also $\binom{n+1}{k}$.

□

Binomial Theorem: $x = 1$

Theorem: $2^n = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{0}$

Proof: How many subsets of $\{1, \dots, n\}$?

Construct a subset with sequence of n choices:

element i is in or is not in the subset: 2 poss.

First rule of counting: $2 \times 2 \cdots \times 2 = 2^n$ subsets.

How many subsets of $\{1, \dots, n\}$?

$\binom{n}{i}$ ways to choose i elts of $\{1, \dots, n\}$.

Sum over i to get total number of subsets..which is also 2^n .

□

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?

How many contain the first element?

Chose first element, need $k-1$ more from remaining n elements.

$\implies \binom{n}{k-1}$

How many don't contain the first element?

Need to choose k elements from remaining n elts.

$\implies \binom{n}{k}$

Sum Rule: size of union of disjoint sets of objects.

Without and with first element \rightarrow disjoint.

So, $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$.

□

Simple Inclusion/Exclusion

Sum Rule: For disjoint sets S and T , $|S \cup T| = |S| + |T|$

Used to reason about all subsets

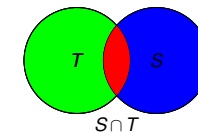
by adding number of subsets of size 1, 2, 3, ...

Also reasoned about subsets that contained

or didn't contain an element. (E.g., first element, first i elements.)

Inclusion/Exclusion Rule:

For any S and T , $|S \cup T| = |S| + |T| - |S \cap T|$.



In T . $\implies |T|$

In S . $\implies + |S|$

Elements in $S \cap T$ are counted twice.

Subtract. $\implies -|S \cap T|$

$$|S \cup T| = |S| + |T| - |S \cap T|$$

Simple Inclusion/Exclusion

Sum Rule: For disjoint sets S and T , $|S \cup T| = |S| + |T|$

Used to reason about all subsets

by adding number of subsets of size 1, 2, 3, ...

Also reasoned about subsets that contained or didn't contain an element. (E.g., first element, first i elements.)

Inclusion/Exclusion Rule: For any S and T ,

$$|S \cup T| = |S| + |T| - |S \cap T|.$$

Example: How many 10-digit phone numbers have 7 as their first or second digit?

S = phone numbers with 7 as first digit. $|S| = 10^9$

T = phone numbers with 7 as second digit. $|T| = 10^9$.

$S \cap T$ = phone numbers with 7 as first and second digit. $|S \cap T| = 10^8$.

Answer: $|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$.

Summary.

First Rule of counting: Objects from a sequence of choices:
 n_i possibilities for i th choice : $n_1 \times n_2 \times \dots \times n_k$ objects.

Second Rule of counting: If order does not matter.
 Count with order: Divide number of orderings. Typically: $\binom{n}{k}$.

Stars and Bars: Sample k objects with replacement from n .
 Order doesn't matter: Typically: $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$.

Inclusion/Exclusion: two sets of objects.
 Add number of each subtract intersection of sets.

Sum Rule: If disjoint just add.

Combinatorial Proofs: Identity from counting same in two ways.

Pascal's Triangle Example: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

RHS: Number of subsets of $n+1$ items size k .

LHS: $\binom{n}{k-1}$ counts subsets of $n+1$ items with first item.

$\binom{n}{k}$ counts subsets of $n+1$ items without first item.

Disjoint - so add!

Inclusion/Exclusion

$$|A_1 \cup \dots \cup A_n| =$$

$$\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \dots (-1)^{n+1} |A_1 \cap \dots \cap A_n|.$$

Idea: For $n=3$ how many times is an element counted?

Consider $x \in A_i \cap A_j$.

x counted once for $|A_i|$ and once for $|A_j|$.

x subtracted from count once for $|A_i \cap A_j|$.

Total: $2 - 1 = 1$.

Consider $x \in A_1 \cap A_2 \cap A_3$

x counted once in each term: $|A_1|, |A_2|, |A_3|$.

x subtracted once in terms: $|A_1 \cap A_3|, |A_1 \cap A_2|, |A_2 \cap A_3|$.

x added once in $|A_1 \cap A_2 \cap A_3|$.

Total: $3 - 3 + 1 = 1$.

Formulaically: x is in intersection of three sets.

3 for terms of form $|A_i|$, $\binom{3}{2}$ for terms of form $|A_i \cap A_j|$.

$\binom{3}{3}$ for terms of form $|A_i \cap A_j \cap A_k|$.

Total: $\binom{3}{1} - \binom{3}{2} + \binom{3}{3} = 1$.

Inclusion/Exclusion

$$|A_1 \cup \dots \cup A_n| =$$

$$\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \dots (-1)^{n+1} |A_1 \cap \dots \cap A_n|.$$

Idea: how many times is each element counted?

Element x in m sets: $x \in A_{i_1} \cap A_{i_2} \dots \cap A_{i_m}$.

Counted $\binom{m}{i}$ times in i th summation.

Total: $\binom{m}{1} - \binom{m}{2} + \binom{m}{3} \dots + (-1)^{m-1} \binom{m}{m}$.

Binomial Theorem:

$$(x+y)^m = \binom{m}{0} x^m + \binom{m}{1} x^{m-1} y + \binom{m}{2} x^{m-2} y^2 + \dots + \binom{m}{m} y^m.$$

Proof: m factors in product: $(x+y)(x+y) \dots (x+y)$.

Get a term $x^{m-i} y^i$ by choosing i factors to use for y .
 are $\binom{m}{i}$ ways to choose factors where y is provided. \square

For $x=1, y=-1$,

$$0 = (1-1)^m = \binom{m}{0} - \binom{m}{1} + \binom{m}{2} \dots + (-1)^m \binom{m}{m}$$

$$\implies 1 = \binom{m}{0} = \binom{m}{1} - \binom{m}{2} \dots + (-1)^{m-1} \binom{m}{m}.$$

Each element counted once!

Poll: How big is infinity?

Mark what's true.

- (A) There are more real numbers than natural numbers.
- (B) There are more rational numbers than natural numbers.
- (C) There are more integers than natural numbers.
- (D) pairs of natural numbers \gg natural numbers.

Same Size. Poll.

Two sets are the same size?

- (A) Bijection between the sets.
- (B) Count the objects and get the same number. same size.
- (C) Counting to infinity is hard.

- (A), (B).
- (C)?

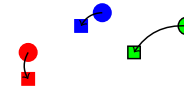
Next up: how big is infinity.

- ▶ Countable
- ▶ Countably infinite.
- ▶ Enumeration

How big are the reals or the integers?

Infinite!
Is one bigger or smaller?

Same size?



Same number?

Make a function $f : \text{Circles} \rightarrow \text{Squares}$.

$f(\text{red circle}) = \text{red square}$

$f(\text{blue circle}) = \text{blue square}$

$f(\text{circle with black border}) = \text{square with black border}$

One to one. Each circle mapped to different square.

One to One: For all $x, y \in D, x \neq y \implies f(x) \neq f(y)$.

Onto. Each square mapped to from some circle .

Onto: For all $s \in R, \exists c \in D, s = f(c)$.

Isomorphism principle: If there is $f : D \rightarrow R$ that is one to one and onto, then, $|D| = |R|$.

Isomorphism principle.

Given a function, $f : D \rightarrow R$.

One to One:

For all $\forall x, y \in D, x \neq y \implies f(x) \neq f(y)$.

or

$\forall x, y \in D, f(x) = f(y) \implies x = y$.

Onto: For all $y \in R, \exists x \in D, y = f(x)$.

$f(\cdot)$ is a **bijection** if it is one to one and onto.

Isomorphism principle:

If there is a bijection $f : D \rightarrow R$ then $|D| = |R|$.

Countable.

How to count?

0, 1, 2, 3, ...

The Counting numbers.

The natural numbers! N

Definition: S is **countable** if there is a bijection between S and some subset of N .

If the subset of N is finite, S has finite **cardinality**.

If the subset of N is infinite, S is **countably infinite**.

Where's 0?

Which is bigger?

The positive integers, \mathbb{Z}^+ , or the natural numbers, \mathbb{N} .

Natural numbers. 0, 1, 2, 3, ...

Positive integers. 1, 2, 3, ...

Where's 0?

More natural numbers!

Consider $f(z) = z - 1$.

For any two $z_1 \neq z_2 \implies z_1 - 1 \neq z_2 - 1 \implies f(z_1) \neq f(z_2)$.

One to one!

For any natural number n , for $z = n + 1$, $f(z) = (n + 1) - 1 = n$.

Onto for \mathbb{N}

Bijection! $\implies |\mathbb{Z}^+| = |\mathbb{N}|$.

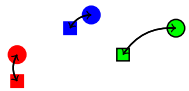
But.. but Where's zero? "Comes from 1."

A bijection is a bijection.

Notice that there is a bijection between N and Z^+ as well.

$$f(n) = n + 1. \quad 0 \rightarrow 1, 1 \rightarrow 2, \dots$$

Bijection from A to $B \implies$ a bijection from B to A .



Inverse function!

Can prove equivalence either way.

Bijection to or from natural numbers implies countably infinite.

Listings..

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if } n \text{ odd.} \end{cases}$$

Another View:

n	$f(n)$
0	0
1	-1
2	1
3	-2
4	2
...	...

Notice that: A listing "is" a bijection with a subset of natural numbers.

Function \equiv "Position in list."

If finite: bijection with $\{0, \dots, |S| - 1\}$

If infinite: bijection with N .

More large sets.

E - Even natural numbers?

$$f: N \rightarrow E.$$

$$f(n) \rightarrow 2n.$$

Onto: $\forall e \in E, f(e/2) = e$. $e/2$ is natural since e is even

One-to-one: $\forall x, y \in N, x \neq y \implies 2x \neq 2y. \equiv f(x) \neq f(y)$

Evens are countably infinite.

Evens are same size as all natural numbers.

All integers?

What about Integers, Z ?

Define $f: N \rightarrow Z$.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if } n \text{ odd.} \end{cases}$$

One-to-one: For $x \neq y$

if x is even and y is odd,

then $f(x)$ is nonnegative and $f(y)$ is negative $\implies f(x) \neq f(y)$

if x is even and y is even,

then $x/2 \neq y/2 \implies f(x) \neq f(y)$

....

Onto: For any $z \in Z$,

if $z \geq 0$, $f(2z) = z$ and $2z \in N$.

if $z < 0$, $f(2|z| - 1) = z$ and $2|z| - 1 \in N$.

Integers and naturals have same size!

Enumerability \equiv countability.

Enumerating (listing) a set implies that it is countable.

"Output element of S ",

"Output next element of S "

...

Any element x of S has *specific, finite* position in list.

$Z = \{0, 1, -1, 2, -2, \dots\}$

$Z = \{\{0, 1, 2, \dots\} \text{ and then } \{-1, -2, \dots\}\}$

When do you get to -1 ? at infinity?

Need to be careful.

61A \equiv streams! Not Sp20/Fa20.

Countably infinite subsets.

Enumerating a set implies countable.

Corollary: Any subset T of a countable set S is countable.

Enumerate T as follows:

Get next element, x , of S ,

output only if $x \in T$.

Implications:

Z^+ is countable.

It is infinite since the list goes on.

There is a bijection with the natural numbers.

So it is countably infinite.

All countably infinite sets have the same cardinality.

Enumeration example.

All binary strings.

$B = \{0, 1\}^*$.

$B = \{\phi, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \dots\}$.

ϕ is empty string.

For any string, it appears at some position in the list.

If n bits, it will appear before position 2^{n+1} .

Should be careful here.

$B = \{\phi; 0, 00, 000, 0000, \dots\}$

Never get to 1.

More fractions?

Enumerate the rational numbers in order...

$0, \dots, 1/2, \dots$

Where is $1/2$ in list?

After $1/3$, which is after $1/4$, which is after $1/5$...

A thing about fractions:

any two fractions has another fraction between it.

Can't even get to "next" fraction!

Can't list in "order".

Pairs of natural numbers.

Consider pairs of natural numbers: $N \times N$

E.g.: $(1, 2)$, $(100, 30)$, etc.

For finite sets S_1 and S_2 ,

then $S_1 \times S_2$

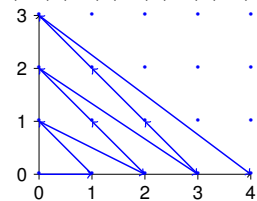
has size $|S_1| \times |S_2|$.

So, $N \times N$ is countably infinite squared ???

Pairs of natural numbers.

Enumerate in list:

$(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2), \dots$



The pair (a, b) , is in first $\approx (a + b + 1)(a + b)/2$ elements of list!
(i.e., "triangle").

Countably infinite.

Same size as the natural numbers!!

Poll.

Enumeration to get bijection with naturals?

(A) Integers: First all negatives, then positives.

(B) Integers: By absolute value, break ties however.

(C) Pairs of naturals: by sum of values, break ties however.

(D) Pairs of naturals: by value of first element.

(E) Pairs of integers: by sum of values, break ties.

(F) Pairs of integers: by sum of absolute values, break ties.

(B),(C), (F).

Rationals?

Positive rational number.

Lowest terms: a/b

$a, b \in N$

with $\gcd(a, b) = 1$.

Infinite subset of $N \times N$.

Countably infinite!

All rational numbers?

Negative rationals are countable. (Same size as positive rationals.)

Put all rational numbers in a list.

First negative, then nonnegative ??? No!

Repeatedly and alternatively take one from each list.

Interleave Streams in 61A

The rationals are countably infinite.

Real numbers..

Real numbers are same size as integers?

The reals.

Are the set of reals countable?

Lets consider the reals $[0, 1]$.

Each real has a decimal representation.

.500000000... $(1/2)$

.785398162... $\pi/4$

.367879441... $1/e$

.632120558... $1 - 1/e$

.345212312... Some real number

Diagonalization.

If countable, there a listing, L contains all reals. For example

0: .500000000...

1: .785398162...

2: .367879441...

3: .632120558...

4: .345212312...

⋮

Construct "diagonal" number: .77677...

Diagonal Number: Digit i is 7 if number i 's i th digit is not 7
and 6 otherwise.

Diagonal number for a list differs from every number in list!

Diagonal number not in list.

Diagonal number is real.

Contradiction!

Subset $[0, 1]$ is not countable!!

Poll: diagonalization Proof.

Mark parts of proof.

- (A) Integers are larger than naturals cuz obviously.
 - (B) Integers are countable cuz, interleaving bijection.
 - (C) Reals are uncountable cuz obviously!
 - (D) Reals can't be in a list: diagonal number not on list.
- (B), (C)?, (D)