CS 70 Spring 2022 — Discrete Mathematics and Probability Theory — Koushik Sen and Satish Rao — HW 11

Due: Saturday 4/9, 4:00 PM
Grace period until Saturday 4/9, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Class Enrollment

Lydia has just started her CalCentral enrollment appointment. She needs to register for a marine science class and CS 70. There are no waitlists, and she can attempt to enroll once per day in either class or both. The CalCentral enrollment system is strange and picky, so the probability of enrolling successfully in the marine science class on each attempt is μ and the probability of enrolling successfully in CS 70 on each attempt is λ. Also, these events are independent.

(a) Suppose Lydia begins by attempting to enroll in the marine science class everyday and gets enrolled in it on day M. What is the distribution of M?

(b) Suppose she is not enrolled in the marine science class after attempting each day for the first 5 days. What is $\mathbb{P}[M = i | M > 5]$, the conditional distribution of M given $M > 5$?

(c) Once she is enrolled in the marine science class, she starts attempting to enroll in CS 70 from day $M+1$ and gets enrolled in it on day C. Find the expected number of days it takes Lydia to enroll in both the classes, i.e. $\mathbb{E}[C]$.

Suppose instead of attempting one by one, Lydia decides to attempt enrolling in both the classes from day 1. Let M be the number of days it takes to enroll in the marine science class, and C be the number of days it takes to enroll in CS 70.

(d) What is the distribution of M and C now? Are they independent?

(e) Let X denote the day she gets enrolled in her first class and let Y denote the day she gets enrolled in both the classes. What is the distribution of X?

(f) What is the expected number of days it takes Lydia to enroll in both classes now, i.e. $\mathbb{E}[Y]$.

(g) What is the expected number of classes she will be enrolled in by the end of 14 days?

2 Geometric and Poisson

Let $X \sim \text{Geo}(p)$ and $Y \sim \text{Poisson}(\lambda)$ be independent. random variables. Compute $\mathbb{P}[X > Y]$. Your final answer should not have summations.

Hint: Use the total probability rule.

3 Two Sides of a Coin

(a) Alice has 1 fair coin. She tosses the coin until she sees both sides. In expectation, how many tosses does this take?

(b) Bob has 2 fair coins. He tosses the first coin until he sees both sides of it, then tosses the second coin until he sees both sides of it. In expectation, how many total tosses does this take?

(c) Charlie has 2 fair coins. He repeatedly tosses the pair of coins simultaneously (i.e., two tosses at a time), until he has seen both sides of both coins. In expectation, how many total tosses does this take?

4 Swaps and Cycles

We'll say that a permutation $\pi = (\pi(1), \dots, \pi(n))$ contains a *swap* if there exist $i, j \in \{1, \dots, n\}$ so that $\pi(i) = j$ and $\pi(j) = i$, where $i \neq j$.

(a) What is the expected number of swaps in a random permutation?

(b) What about the variance?

(c) In the same spirit as above, we'll say that π contains a *s-cycle* if there exist $i_1, \dots, i_s \in \{1, \dots, n\}$ with $\pi(i_1) = i_2, \pi(i_2) = i_3, \dots, \pi(i_s) = i_1$. Compute the expectation of the number of s-cycles.

5 Will I Get My Package?

A delivery guy in some company is out delivering n packages to n customers, where n is a natural number greater than 1. Not only does he hand each customer a package uniformly at random from the remaining packages, he opens the package before delivering it with probability $1/2$. Let X be the number of customers who receive their own packages unopened.

(a) Compute the expectation $\mathbb{E}[X]$.

(b) Compute the variance $\text{Var}(X)$.

6 Diversify Your Hand

You are dealt 5 cards from a standard 52 card deck. Let X be the number of distinct values in your hand. For instance, the hand (A, A, A, 2, 3) has 3 distinct values.

(a) Calculate $\mathbb{E}[X]$.

(b) Calculate $\text{Var}(X)$.

7 Double-Check Your Intuition Again

(a) You roll a fair six-sided die and record the result X. You roll the die again and record the result Y.

(i) What is $\text{cov}(X+Y, X-Y)$?

(ii) Prove that $X+Y$ and $X-Y$ are not independent.

For each of the problems below, if you think the answer is "yes" then provide a proof. If you think the answer is "no", then provide a counterexample.

(b) If X is a random variable and $\text{Var}(X) = 0$, then must X be a constant?

(c) If X is a random variable and c is a constant, then is $\text{Var}(cX) = c\,\text{Var}(X)$?

(d) If A and B are random variables with nonzero standard deviations and $\text{Corr}(A,B) = 0$, then are A and B independent?

(e) If X and Y are not necessarily independent random variables, but $\text{Corr}(X,Y) = 0$, and X and Y have nonzero standard deviations, then is $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$?

(f) If X and Y are random variables then is $\mathbb{E}[\max(X,Y)\min(X,Y)] = \mathbb{E}[XY]$?

(g) If X and Y are independent random variables with nonzero standard deviations, then is

$$\text{Corr}(\max(X,Y), \min(X,Y)) = \text{Corr}(X,Y)?$$