CS 70 Spring 2022 — Discrete Mathematics and Probability Theory — Koushik Sen and Satish Rao — HW 10

Due: Saturday 4/2, 4:00 PM
Grace period until Saturday 4/2, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Independent Complements

Let Ω be a sample space, and let $A, B \subseteq \Omega$ be two independent events.

(a) Prove or disprove: $\overline{A}$ and $\overline{B}$ must be independent.

(b) Prove or disprove: A and $\overline{B}$ must be independent.

(c) Prove or disprove: A and $\overline{A}$ must be independent.

(d) Prove or disprove: It is possible that $A = B$.

2 Symmetric Marbles

A bag contains 4 red marbles and 4 blue marbles. Leanne and Sylvia play a game where they draw four marbles in total, one by one, uniformly at random, without replacement. Leanne wins if there are more red than blue marbles, and Sylvia wins if there are more blue than red marbles. If there are an equal number of marbles, the game is tied.

(a) Let A_1 be the event that the first marble is red and let A_2 be the event that the second marble is red. Are A_1 and A_2 independent?

(b) What is the probability that Leanne wins the game?

(c) Given that Leanne wins the game, what is the probability that all of the marbles were red?

Now, suppose the bag contains 8 red marbles and 4 blue marbles. Moreover, if there are an equal number of red and blue marbles among the four drawn, Leanne wins if the third marble is red, and Sylvia wins if the third marble is blue.

(d) What is the probability that the third marble is red?

(e) Given that there are k red marbles among the four drawn, where $0 \le k \le 4$, what is the probability that the third marble is red? Answer in terms of k.

(f) Given that the third marble is red, what is the probability that Leanne wins the game?

3 Cliques in Random Graphs

Consider the graph $G = (V, E)$ on n vertices which is generated by the following random process: for each pair of vertices u and v, we flip a fair coin and place an (undirected) edge between u and v if and only if the coin comes up heads.

(a) What is the size of the sample space?

(b) A k-clique in graph is a set S of k vertices which are pairwise adjacent (every pair of vertices is connected by an edge). For example a 3-clique is a triangle. Let's call the event that S forms a clique E_S. What is the probability of E_S for a particular set S of k vertices?

(c) Suppose that $V_1 = \{v_1, \ldots, v_\ell\}$ and $V_2 = \{w_1, \ldots, w_k\}$ are two arbitrary sets of vertices. What conditions must V_1 and V_2 satisfy in order for E_{V_1} and E_{V_2} to be independent? Prove your answer.

(d) Prove that $\binom{n}{k} \le n^k$. (You might find this useful in part (e))

(e) Prove that the probability that the graph contains a k-clique, for $k \ge 4\log_2 n + 1$, is at most $1/n$.

4 (Un)conditional (In)equalities

Let us consider a sample space $\Omega = \{\omega_1, \ldots, \omega_N\}$ of size $N > 2$ and two probability functions $\mathbb{P}_1$ and $\mathbb{P}_2$ on it. That is, we have two probability spaces: $(\Omega, \mathbb{P}_1)$ and $(\Omega, \mathbb{P}_2)$.

(a) Suppose that for every subset $A \subseteq \Omega$ of size $|A| = 2$ and for every outcome $\omega \in \Omega$, it is true that $\mathbb{P}_1[\omega \mid A] = \mathbb{P}_2[\omega \mid A]$.

  (i) Let $A = \{\omega_i, \omega_j\}$ for some $i, j \in \{1, \ldots, N\}$. What can you say about $\frac{\mathbb{P}_1[\omega_i]}{\mathbb{P}_1[\omega_j]}$ and $\frac{\mathbb{P}_2[\omega_i]}{\mathbb{P}_2[\omega_j]}$?

  (ii) Is it necessarily true that $\mathbb{P}_1[\omega] = \mathbb{P}_2[\omega]$ for all $\omega \in \Omega$? That is, if $\mathbb{P}_1$ and $\mathbb{P}_2$ are equal conditional on events of size 2, are they equal unconditionally? (*Hint*: Remember that probabilities must add up to 1.)

(b) Suppose that for every subset $A \subseteq \Omega$ of size $|A| = k$, where k is some fixed element in $\{2, \ldots, N\}$, and for every outcome $\omega \in \Omega$, it is true that $\mathbb{P}_1[\omega \mid A] = \mathbb{P}_2[\omega \mid A]$. Is it necessarily true that $\mathbb{P}_1[\omega] = \mathbb{P}_2[\omega]$ for all $\omega \in \Omega$? (*Hint*: Use part (a).)

For the following two parts, assume that $\Omega = \left\{(a_1,\ldots,a_k) \mid \sum_{j=1}^{k} a_j = n\right\}$ is the set of configurations of n balls into k labeled bins, and let $\mathbb{P}_1$ be the probabilities assigned to these configurations by throwing the balls independently one after another and they will land into any of the k bins uniformly at random, and let $\mathbb{P}_2$ be the probabilities assigned to these configurations by uniformly sampling one of these configurations.

As an example, suppose $k = 6$ and $n = 2$. $\mathbb{P}_1$ is equivalent to rolling 2 six-sided dice, and letting a_i be the number of is that appear. $\mathbb{P}_2$ is equivalent to sampling uniformly from all unordered pairs (i, j) with $1 \le i, j \le 6$.

(c) Let A be the event that all n balls are in exactly one bin.

  (i) What are $\mathbb{P}_1[\omega \mid A]$ and $\mathbb{P}_2[\omega \mid A]$ for any $\omega \in A$?

  (ii) Repeat part (i) for $\omega \in \Omega \setminus A$.

  (iii) Is it true that $\mathbb{P}_1[\omega] = \mathbb{P}_2[\omega]$ for all $\omega \in \Omega$?

(d) For the special case of $n = 9$ and $k = 3$, provide two outcomes B and C, so that $\mathbb{P}_1[B] < \mathbb{P}_2[B]$ and $\mathbb{P}_1[C] > \mathbb{P}_2[C]$. Provide justification.

5 Cookie Jars

You have two jars of cookies, each of which starts with n cookies initially. Every day, when you come home, you pick one of the two jars randomly (each jar is chosen with probability $1/2$) and eat one cookie from that jar. One day, you come home and reach inside one of the jars of cookies, but you find that is empty! Let X be the random variable representing the number of remaining cookies in non-empty jar at that time. What is the distribution of X?

6 Maybe Lossy Maybe Not

Let us say that Alice would like to send a message to Bob, over some channel. Alice has a message of length 4.

(a) Packets are dropped with probability p. If Alice sends 5 packets, what is probability that Bob can successfully reconstruct Alice's message using polynomial interpolation?

(b) Again, packets can be dropped with probability p. The channel may additionally corrupt 1 packet after deleting packets. Alice realizes this and sends 8 packets for a message of length 4. What is the probability that Bob receives enough packets to successfully reconstruct Alice's message using Berlekamp-Welch?

(c) Again, packets can be dropped with probability p. This time, packets may be corrupted with probability q. A packet being dropped is independent of whether or not is corrupted (i.e. a packet may be both corrupted and dropped). Consider the original scenario where Alice sends 5 packets for a message of length 4. What is probability that Bob can correctly reconstruct Alice's message using polynomial interpolation on all of the points he receives?