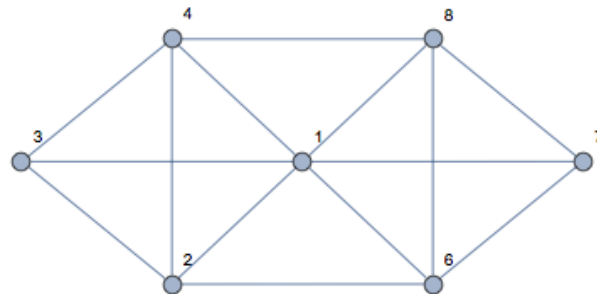


1 Optimal Candidates

In the notes, we proved that the propose-and-reject algorithm always outputs the job-optimal pairing. However, we never explicitly showed why it is guaranteed that putting every job with its optimal candidate results in a pairing at all. Prove by contradiction that no two jobs can have the same optimal candidate. (Note: your proof should not rely on the fact that the propose-and-reject algorithm outputs the job-optimal pairing.)

2 Eulerian Tour and Eulerian Walk



- (a) Is there an Eulerian tour in the graph above? If no, give justification. If yes, provide an example.
- (b) Is there an Eulerian walk in the graph above? An Eulerian walk is a walk that uses each edge exactly once. If no, give justification. If yes, provide an example.

(c) What is the condition that there is an Eulerian walk in an undirected graph? Briefly justify your answer.

3 Not everything is normal: Odd-Degree Vertices

Claim: Let $G = (V, E)$ be an undirected graph. The number of vertices of G that have odd degree is even.

Prove the claim above using:

- (i) Direct proof (e.g., counting the number of edges in G). *Hint: in lecture, we proved that $\sum_{v \in V} \deg v = 2|E|$.*

(ii) Induction on $m = |E|$ (number of edges)

(iii) Induction on $n = |V|$ (number of vertices)

4 Coloring Trees

Prove that all trees with at least 2 vertices are *bipartite*: the vertices can be partitioned into two groups so that every edge goes between the two groups.

[*Hint*: Use induction on the number of vertices.]