Today.

Quick review.

Finish Graphs (maybe.)

Proof of “handshake” lemma.

Lemma: The sum of degrees is $2|E|$, for a graph $G = (V, E)$.
What’s true?

(A) The number of edge-vertex incidences for an edge e is 2.
(B) The total number of edge-vertex incidences is $|V|$.
(C) The total number of edge-vertex incidences is $2|E|$.
(D) The number of edge-vertex incidences for a vertex v is its degree.
(E) The sum of degrees is $2|E|$.
(F) Total number of edge-vertex incidences is sum of vertex degrees.

(B) is false. The others are statements in the proof.

Poll: Euler concepts.

A graph is Eueleurian if it is connected and has even degree.

A graph is Eulerian if it is conncected and has a tour that uses every edge once.

Mark correct statements for a connected graph where all vertices have even degree. (Here a tour means uses an edge exactly once, but may involve a vertex several times.

(A) There is no Hotel California in this graph.
(B) Walking on unused edges, starting at v, eventually “stuck” at v.
(C) Removing a tour leaves a graph of even degree.
(D) Removing a tour leaves a connected graph.
(E) Remove set of edges E' in connected graph, connected component is incident to edge in E'
(F) A tour connecting a set of connected components, each with a Eulerian tour is really cool! This implies the graph is Eulerian.

Only (D) is false. The rest are steps in the proof.

Lecture 6.

Euler’s Formula.

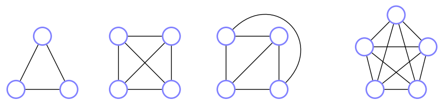
Planar Six and then Five Color theorem.

Types of graphs.

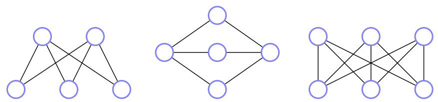
- Complete Graphs.
- Trees (a little more.)
- Hypercubes.

Planar graphs.

A graph that can be drawn in the plane without edge crossings.

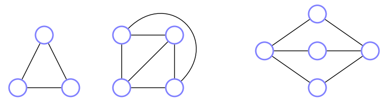


Planar? Yes for Triangle.
Four node complete? Yes.
(complete $\equiv$ every edge present. K_n is n-vertex complete graph.)
Five node complete or K_5 ? No! Why? Later.



Two to three nodes, bipartite? Yes.
Three to three nodes, complete/bipartite or $K_{3,3}$. No. Why? Later.

Euler’s Formula.



Faces: connected regions of the plane.

How many faces for
- triangle? 2
- complete on four vertices or K_4? 4
- bipartite, complete two/three or $K_{2,3}$? 3

v is number of vertices, e is number of edges, f is number of faces.

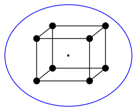
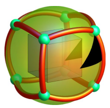
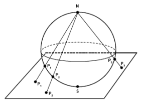
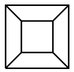
Euler’s Formula: Connected planar graph has $v + f = e + 2$.

Triangle: $3 + 2 = 3 + 2$!
K_4: $4 + 4 = 6 + 2$!
$K_{2,3}$: $5 + 3 = 6 + 2$!

Examples = 3! Proven! Not!!!!

Euler and Polyhedron.

Greeks knew formula for polyhedron.

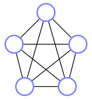
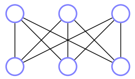
Faces? 6. Edges? 12. Vertices? 8.
Euler: Connected planar graph: $v+f=e+2$.
$8+6=12+2$.

Greeks couldn't prove it. Induction? Remove vertice for polyhedron?
Polyhedron without holes $\equiv$ Planar graphs.

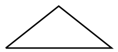
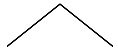
For Convex Polyhedron:
Surround by sphere.
Project from internal point polytope to sphere: drawing on sphere.
Project Sphere-N onto Plane: drawing on plane.

Euler proved formula thousands of years later!

Euler and non-planarity of K_5 and $K_{3,3}$

Euler: $v+f=e+2$ for connected planar graph.
We consider simple graphs where $v \geq 3$.
Consider Face edge Adjacencies with multiplicities

Each face is adjacent to at least three edges($v > 2$).
$\geq 3f$ face-edge adjacencies.
Each edge is adjacent to two faces.
$= 2e$ face-edge adjacencies.
$\Longrightarrow$ $3f \leq 2e$ for any planar graph with $v > 2$. Or $f \leq \frac{2}{3}e$.

Plug into Euler: $v + \frac{2}{3}e \geq e+2 \implies e \leq 3v-6$

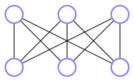
K_5 Edges? $e = 4+3+2+1 = 10$. Vertices? $v = 5$.
$10 \not\leq 3(5)-6 = 9$. $\implies$ K_5 is not planar.

Planar $\implies$ $e \leq 3v-6$. Flow Poll.

Euler's formula: $v+f=e+2$
Consider graph with > 2 vertices. Understand the following.
(A) Every face is incident to ≥ 3 edges.
(B) Face-edge incidences $\geq 3f$
(C) Every edge is incident (with multiplicity) to 2 faces.
(D) Face edge incidences $= 2e$
(E) $3f \leq$ Face-ege-incidence $= 2e$
(F) $3(e+2-v) <= 2e$

Conclusion: $e <= 3v-6$

Proving non-planarity for $K_{3,3}$



$K_{3,3}$? Edges? 9. Vertices. 6.

$e \leq 3(v)-6$ for planar graphs.

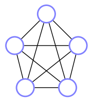
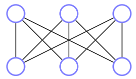
$9 \leq 3(6)-6$? Sure!

Step in proof of K_5: faces are adjacent to ≥ 3 edges.

For $K_{3,3}$ every cycle is of even length or incident ≥ 4 faces.

Finish in homework!

Planarity and Euler

These graphs **cannot** be drawn in the plane without edge crossings.

Euler's Formula: $v+f=e+2$ for any planar drawing.

$\implies$ for simple planar graphs: $e \leq 3v-6$.
Idea: Face is a cycle in graph of length 3.
Count face-edge incidences.

$\implies$ for bipartite simple planar graphs: $e \leq 2v-4$.
Idea: face is a cycle in graph of length 4.
Count face-edge incidences.

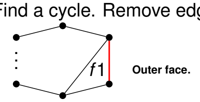
Proved absolutely no drawing can work for these graphs.

So......so ...Cool!

Euler's formula.

Euler: Connected planar graph has $v+f=e+2$.

Proof: Induction on e.
Base: $e=0$, $v=f=1$.
Induction Step:
If it is a tree. $e=v-1$, $f=1$, $v+1=(v-1)+2$. Yes.
If not a tree.
Find a cycle. Remove edge.



Joins two faces.
New graph: v-vertices. $e-1$ edges. $f-1$ faces. Planar.
$v+(f-1)=(e-1)+2$ by induction hypothesis.
Therefore $v+f=e+2$. $\square$

Quick:
$v+1=(v-1)+2$, add edge: $f \to f+1$, $e \to e+1$.

Euler's Proof.Poll.

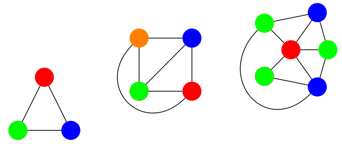
Euler: Connected planar graph has $v + f = e + 2$.
Steps/concepts in proof of euler's formula.

(A) Planar drawing of tree has 1 face.
(B) Tree has $|V| - 1$ edges.
(C) Induction.
(D) face is adjacent to at least 3 edges.
(E) edge has two edge-vertex incidences.
(F) Add edge to planar drawing splits a face.

All are true and all are relevant to the proof, though (E) is more analagous than direct.

Graph Coloring.

Given $G = (V, E)$, a coloring of G assigns colors to vertices V where for each edge the endpoints have different colors.

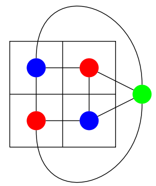


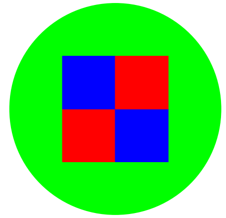
Notice that the last one, has one three colors.
  Fewer colors than number of vertices.
  Fewer colors than max degree node.

Interesting things to do. Algorithm!

Planar graphs and maps.

Planar graph coloring $\equiv$ map coloring.





Four color theorem is about planar graphs!

Six color theorem.

Theorem: Every planar graph can be colored with six colors.

Proof:
Recall: $e \leq 3v - 6$ for any planar graph where $v > 2$.
  From Euler's Formula.

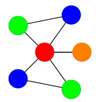
Total degree: $2e$
Average degree: $= \frac{2e}{v} \leq \frac{2(3v-6)}{v} \leq 6 - \frac{12}{v}$.

There exists a vertex with degree < 6 or at most 5.

  Remove vertex v of degree at most 5.
  Inductively color remaining graph.
  Color is available for v since only five neighbors...
    and only five colors are used. □

Five color theorem: prelimnary.

Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.



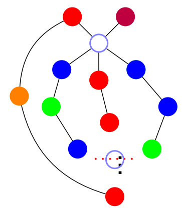
Look at only green and blue.
Connected components.
Can switch in one component.
Or the other.

Five color theorem

Theorem: Every planar graph can be colored with five colors.

Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.

Proof: Again with the degree 5 vertex. Again recurse.



Assume neighbors are colored all differently.
  Otherwise one of 5 colors is available. $\implies$ Done!
Switch green and blue in green's component.
  Done. Unless blue-green path to blue.
Switch orange and red in oranges component.
  Done. Unless red-orange path to red.

Planar. $\implies$ paths intersect at a vertex!

What color is it?
  Must be blue or green to be on that path.
  Must be red or orange to be on that path.

Contradiction. Can recolor one of the neighbors.
Gives an available color for center vertex! □

5 color theorem. Flow poll.

Steps/ideas in 5-color theorem.
(A) There is a degree 5 vertex cuz Euler.
(B) Take subgraph of first and third colors, recolor first components.
(C) If a third's component is different, switched coloring is good.
(D) Subgraph of second and fourth colors, can recolor, recolor second component.
(G) At least one separate component cuz planarity.
(F) Shared color of five neighbors, done.

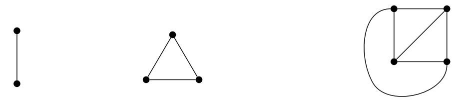
All steps in proof!

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

Proof: Not Today!

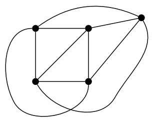
Complete Graph.



K_n complete graph on n vertices.
All edges are present.
Everyone is my neighbor.
Each vertex is adjacent to every other vertex.

How many edges?
Each vertex is incident to $n-1$ edges.
Sum of degrees is $n(n-1) = 2|E|$
$\implies$ Number of edges is $n(n-1)/2$.

K_4 and K_5



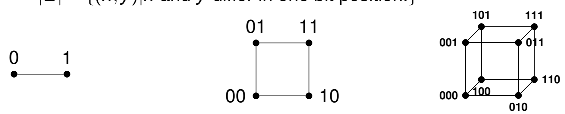
K_5 is not planar.
Cannot be drawn in the plane without an edge crossing!
Prove it! We did!

Hypercubes.

Complete graphs, really connected! But lots of edges.
$|V|(|V|-1)/2$
Trees, few edges. $(|V|-1)$
but just falls apart!

Hypercubes. Really connected. $|V|\log|V|$ edges!
Also represents bit-strings nicely.

$G = (V, E)$
$|V| = \{0,1\}^n$,
$|E| = \{(x,y)|x$ and y differ in one bit position.$\}$

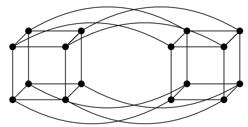


2^n vertices. number of n-bit strings!
$n2^{n-1}$ edges.
2^n vertices each of degree n
total degree is $n2^n$ and half as many edges!

Recursive Definition.

A 0-dimensional hypercube is a node labelled with the empty string of bits.

An n-dimensional hypercube consists of a 0-subcube (1-subcube) which is a $n-1$-dimensional hypercube with nodes labelled $0x$ ($1x$) with the additional edges $(0x, 1x)$.



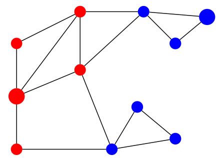
Hypercube: Can't cut me!

Thm: Any subset S of the hypercube where $|S| \leq |V|/2$ has $\geq |S|$ edges connecting it to $V - S$; $|E \cap S \times (V - S)| \geq |S|$

Terminology:
 $(S, V - S)$ is cut.
 $(E \cap S \times (V - S))$ - cut edges.

Restatement: for any cut in the hypercube, the number of cut edges is at least the size of the small side.

Cuts in graphs.



S is red, $V - S$ is blue.

What is size of cut?

Number of edges between red and blue. 4.

Hypercube: any cut that cuts off x nodes has $\geq x$ edges.

Proof of Large Cuts.

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side.
Proof:
Base Case: $n = 1$ V= $\{0,1\}$.
 $S = \{0\}$ has one edge leaving. $|S| = \phi$ has 0.

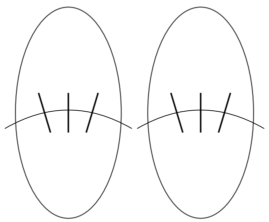
Induction Step Idea

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side.

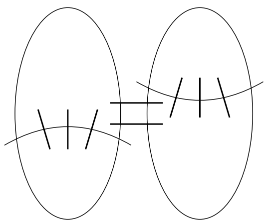
Use recursive definition into two subcubes.

 Two cubes connected by edges.

Case 1: Count edges inside subcube inductively.



Case 2: Count inside and across.



Induction Step

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

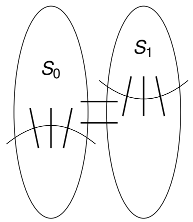
Proof: Induction Step.
Recursive definition:
 $H_0 = (V_0, E_0), H_1 = (V_1, E_1)$, edges E_x that connect them.
 $H = (V_0 \cup V_1, E_0 \cup E_1 \cup E_x)$
 $S = S_0 \cup S_1$ where S_0 in first, and S_1 in other.

Case 1: $|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2$
Both S_0 and S_1 are small sides. So by induction.
 Edges cut in $H_0 \geq |S_0|$.
 Edges cut in $H_1 \geq |S_1|$.

Total cut edges $\geq |S_0| + |S_1| = |S|$. □

Induction Step. Case 2.

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.
Proof: Induction Step. Case 2.



$|S_0| \geq |V_0|/2$.
 Recall Case 1: $|S_0|, |S_1| \leq |V|/2$
$|S_1| \leq |V_1|/2$ since $|S| \leq |V|/2$.
 $\implies \geq |S_1|$ edges cut in E_1.
 $|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2$
 $\implies \geq |V_0| - |S_0|$ edges cut in E_0.

Edges in E_x connect corresponding nodes.
 $\implies = |S_0| - |S_1|$ edges cut in E_x.

Total edges cut:
 $\geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0|$
 $|V_0| = |V|/2 \geq |S|$. □
Also, case 3 where $|S_1| \geq |V|/2$ is symmetric.

Hypercubes and Boolean Functions.

The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on $\{0,1\}^n$.

Central area of study in computer science!

Yes/No Computer Programs $\equiv$ Boolean function on $\{0,1\}^n$

Central object of study.

Summary.

Euler: $v+f=e+2$.
 Tree. Plus adding edge adds face.
Planar graphs: $e \leq 3v = 6$.
 Count face-edge incidences to get $2e \leq 3f$.
 Replace f in Euler.
Coloring:
 degree d vertex can be colored if $d+1$ colors.
 Small degree vertex in planar graph: 6 color theorem.
 Recolor separate and planarity: 5 color theorem.
Graphs:
 Trees: sparsest connected.
 Complete:densest
 Hypercube: middle.

Have a nice weekend!