CS70: Markov Chains.

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1. Examples
2. Definition
3. Hitting Time.
4. Here before there.
5. Stationary Distribution
6. Peridoicity.

Two-State Markov Chain

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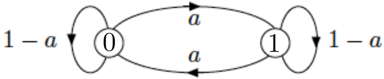

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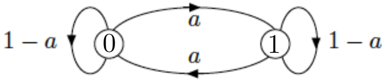

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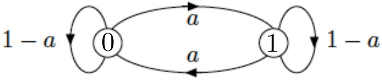

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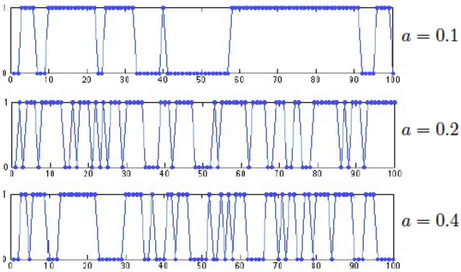

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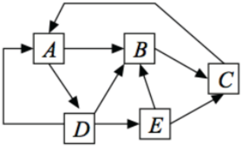
At each step, the MC follows one of the outgoing arrows of the current state, with equal probabilities.



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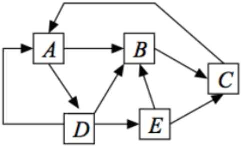

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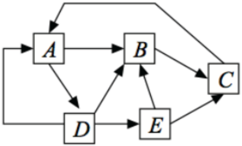

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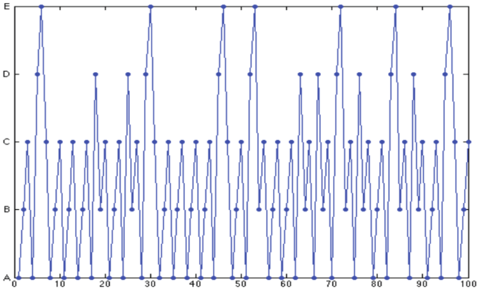

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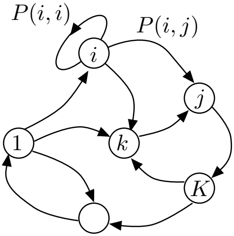

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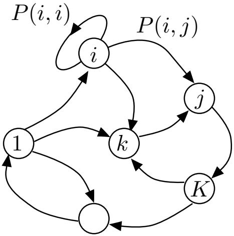

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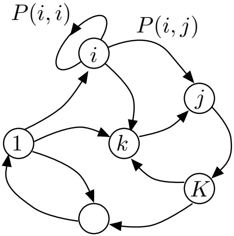

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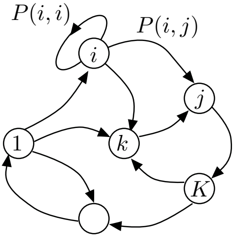

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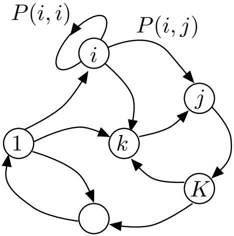

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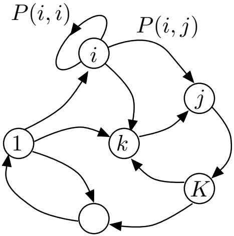

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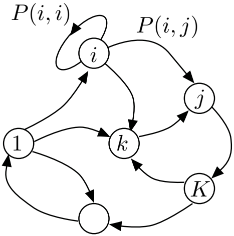

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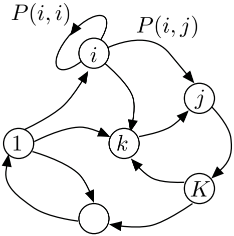

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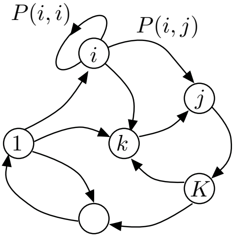

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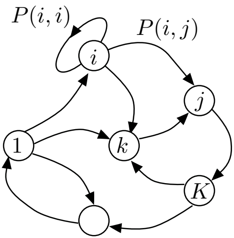

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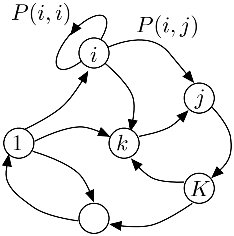

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  $Pr[X_{n+1} = j \mid X_0, \ldots, X_n = i] = P(i,j), i,j \in \mathscr{X}$.

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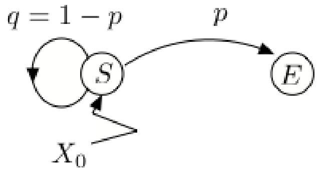

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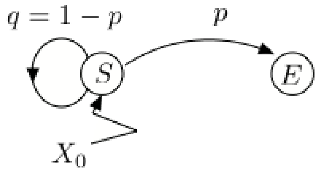

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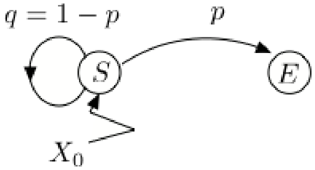

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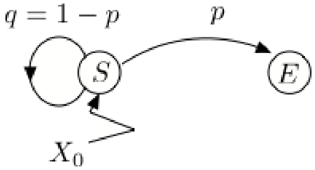

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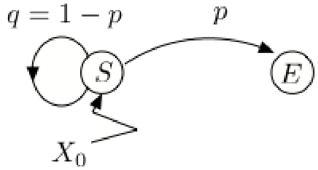

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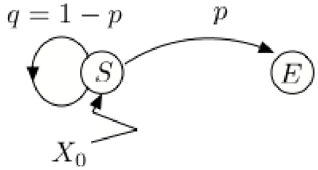

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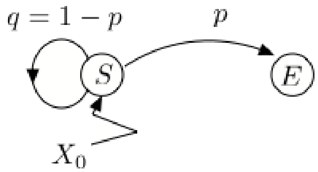

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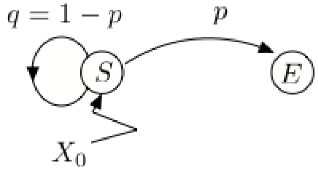

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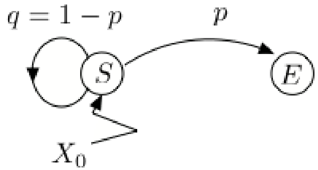

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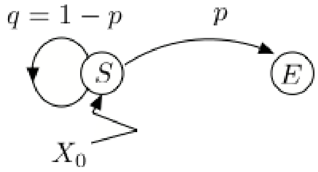

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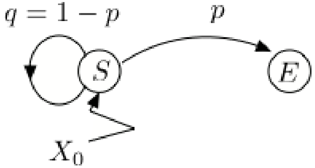

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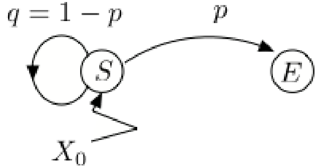

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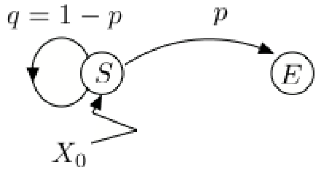

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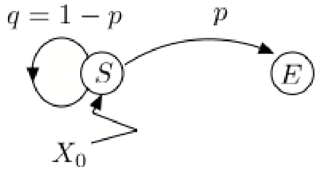

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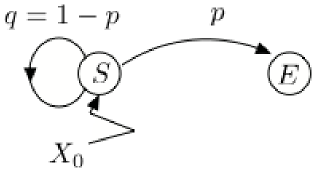

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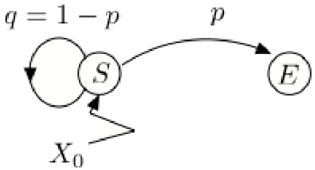

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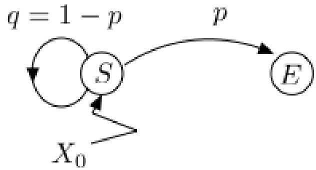

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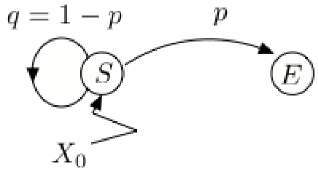

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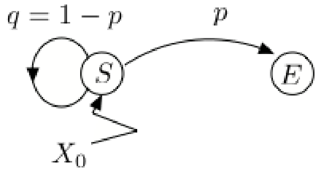

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Justification: N – number of steps until E, starting from S.

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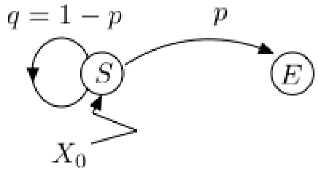

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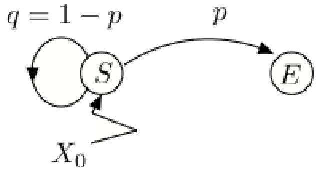

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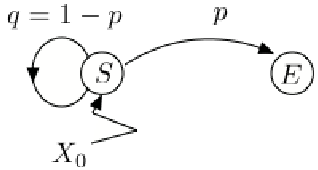

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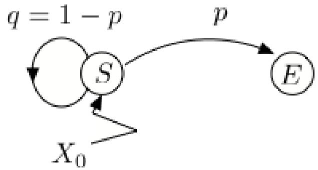

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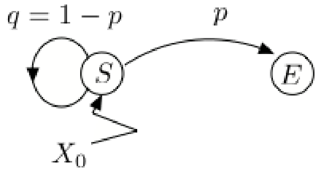

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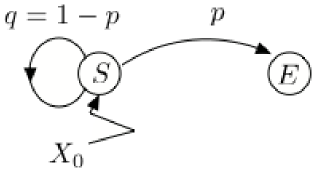

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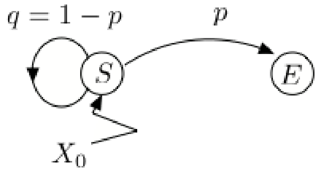

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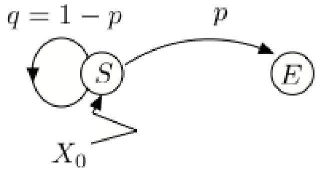

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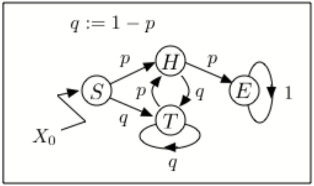

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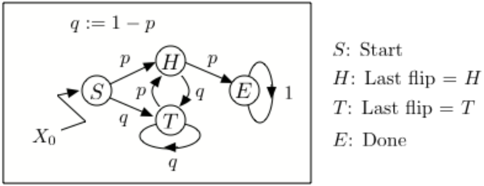

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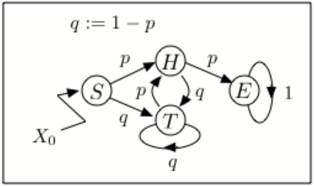

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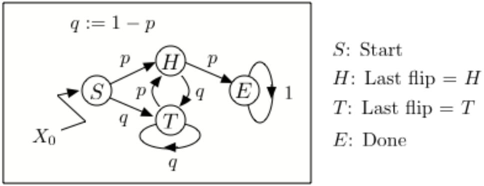

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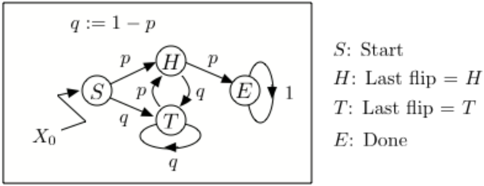

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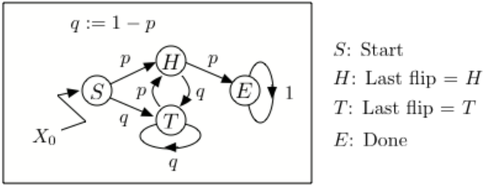

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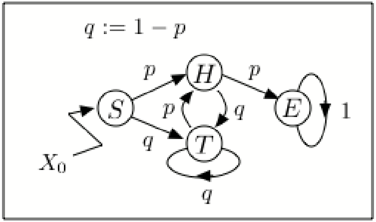

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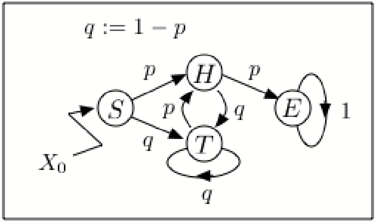

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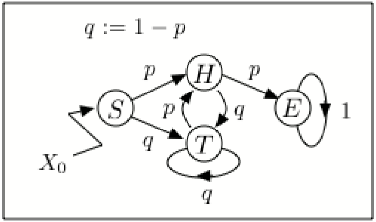

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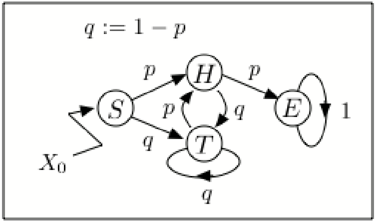

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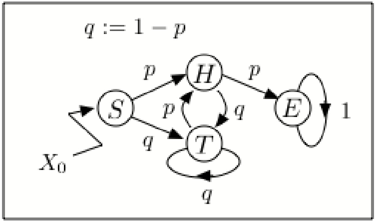

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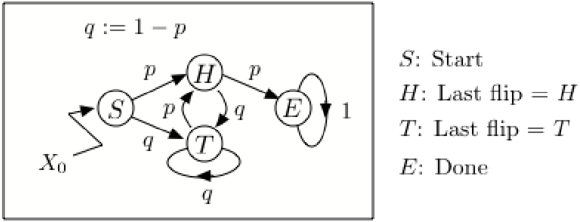

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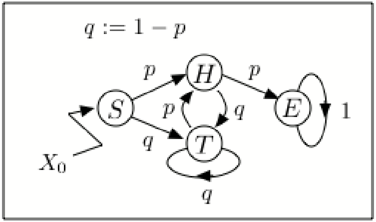

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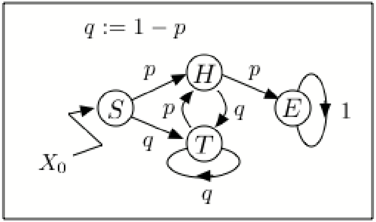

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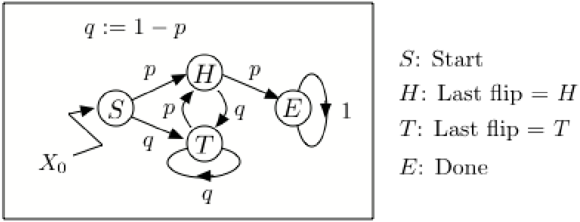

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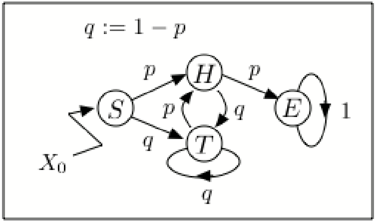

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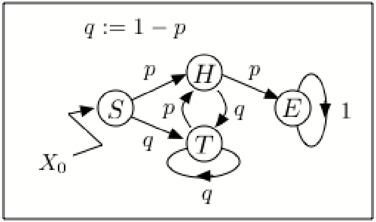

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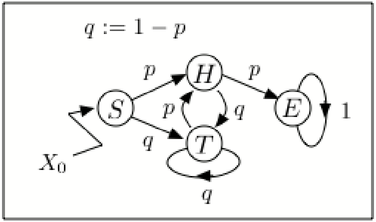

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Let us justify the first step equation for $\beta(T)$.

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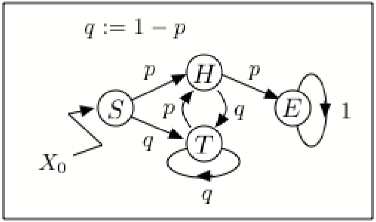

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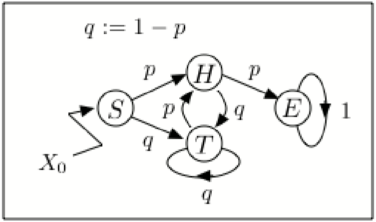

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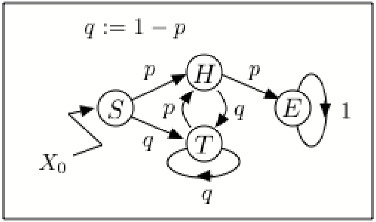

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$N(T)$ – number of steps, starting from T until the MC hits E.
$N(H)$ – be defined similarly.

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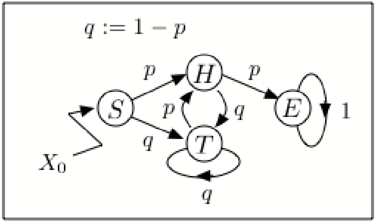

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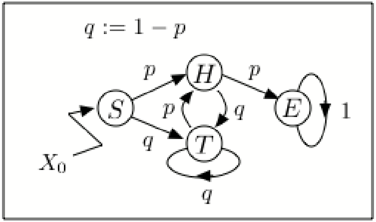

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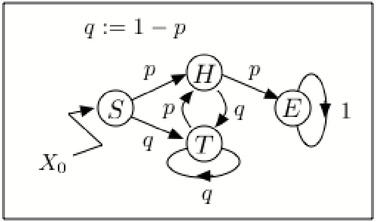

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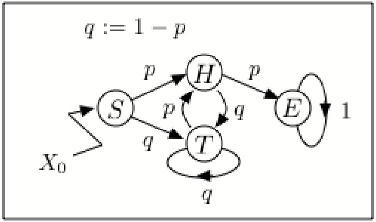

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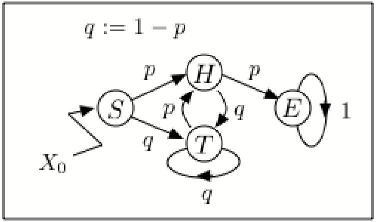

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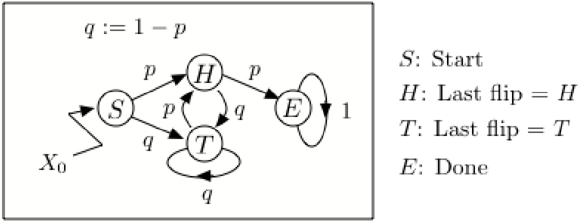

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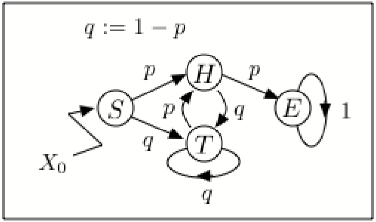

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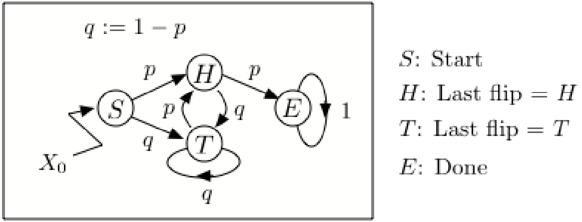

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$$E[N(T)] = 1 + pE[N(H)] + qE[N'(T)],$$

i.e.,

$$\beta(T) = 1 + p\beta(H) + q\beta(T).$$

Hitting Time - Example 3

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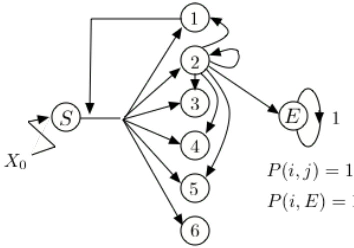

S = Start; E = Done

i = Last roll is i, not done

$P(S, j) = 1/6, j = 1, \ldots, 6$

$P(1, j) = 1/6, j = 1, \ldots, 6$

$P(i, j) = 1/6, i = 2, \ldots, 6; 8 - i \neq j \in \{1, \ldots, 6\}$

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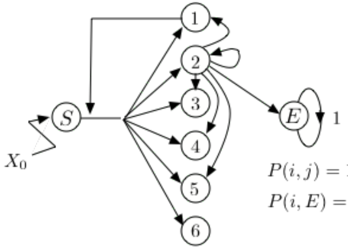

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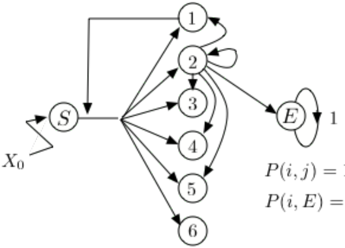

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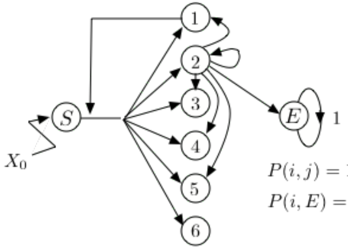

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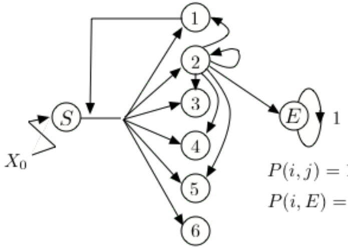

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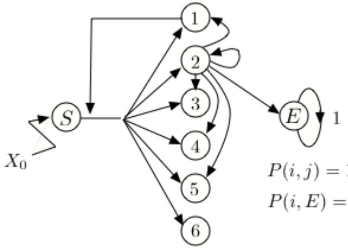

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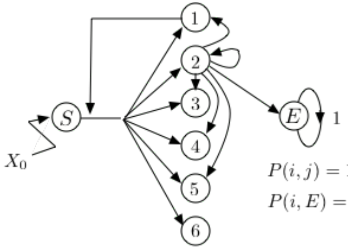

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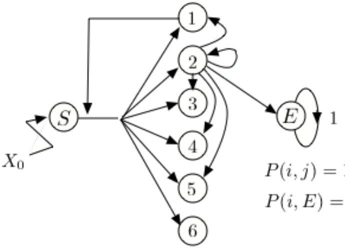

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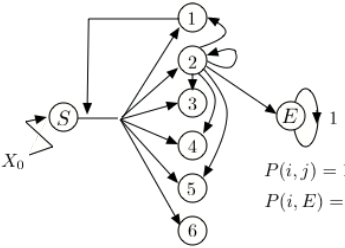

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$$\Rightarrow \cdots \beta(S) = 8.4.$$

Here before There - A before B

Game of “heads or tails” using coin with ‘heads’ probability $p < 0.5$.

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Start with \$10.

Here before There - A before B

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Start with \$10.
Each step, flip yields ‘heads’, earn \$1.

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Each step, flip yields ‘heads’, earn \$1. Otherwise, lose \$1.

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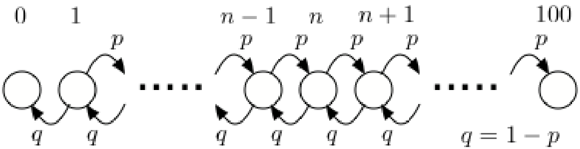

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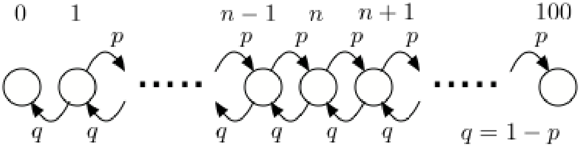

Let $\alpha(n)$ be the probability of reaching 100 before 0, starting from n, for $n = 0, 1, \ldots, 100$.

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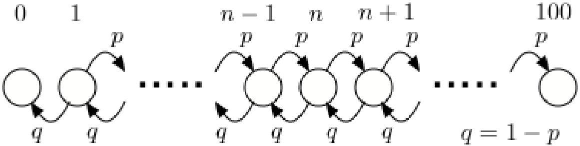

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(A) $\alpha(0) = 0$
(B) $\alpha(0) = 1$.
(C) $\alpha(100) =$

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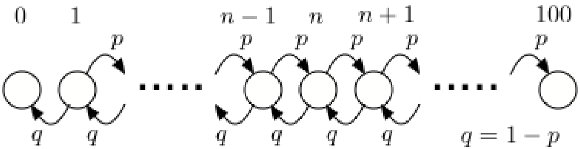

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(D) $\alpha(n) = 1 + p\alpha(n+1) + q\alpha(n-1), 0 < n < 100$.
(E) $\alpha(n) =$

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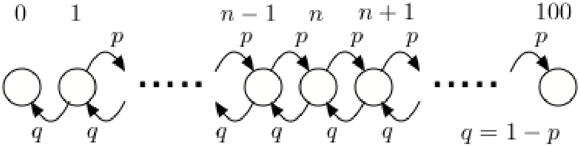

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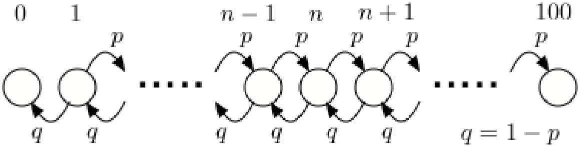
(D) $\alpha(n) = 1 + p\alpha(n+1) + q\alpha(n-1), 0 < n < 100$.
(E) $\alpha(n) = p\alpha(n+1) + q\alpha(n-1), 0 < n < 100$.

Here before There - A before B

Game of “heads or tails” using coin with ‘heads’ probability $p < 0.5$.
Start with \$10.
Each step, flip yields ‘heads’, earn \$1. Otherwise, lose \$1.
What is the probability that you reach \$100 before \$0?



Let $\alpha(n)$ be the probability of reaching 100 before 0, starting from n, for $n = 0, 1, \ldots, 100$.
Which equations are correct?
(A) $\alpha(0) = 0$
(B) $\alpha(0) = 1$.
(C) $\alpha(100) = 1$.
(D) $\alpha(n) = 1 + p\alpha(n+1) + q\alpha(n-1), 0 < n < 100$.
(E) $\alpha(n) = p\alpha(n+1) + q\alpha(n-1), 0 < n < 100$.

(B) is incorrect, 0 is bad.
(D) is incorrect. Confuses expected hitting time with *A* before *B*.

Here before There - A before B

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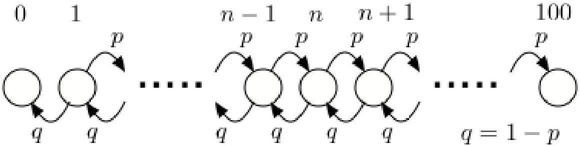

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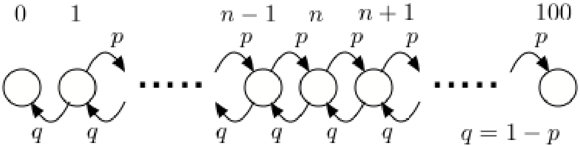

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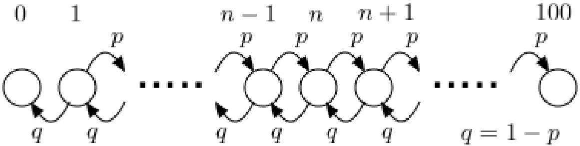

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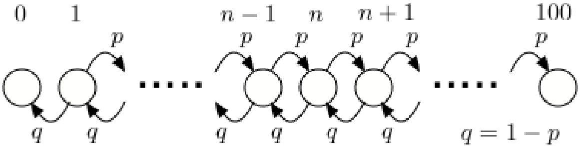

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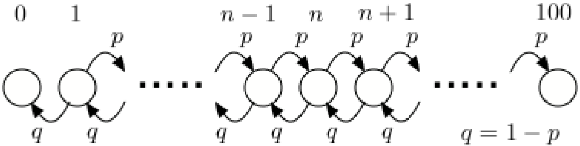

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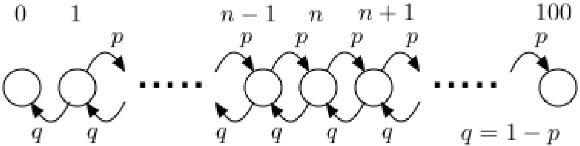

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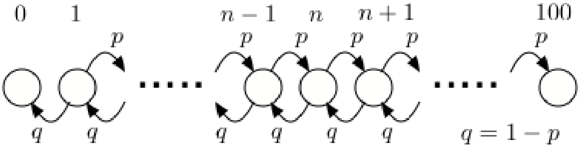

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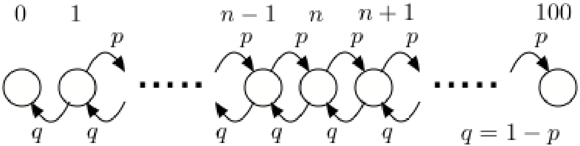

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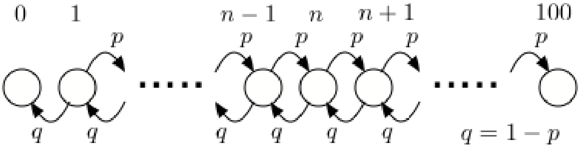

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$$\Rightarrow \alpha(n) = \frac{1-\rho^n}{1-\rho^{100}} \text{ with } \rho = qp^{-1}.$$

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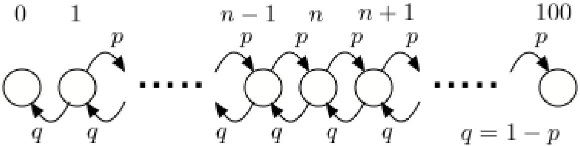

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Here before There - A before B

Game of "heads or tails" using coin with 'heads' probability $p = .48$.

Here before There - A before B

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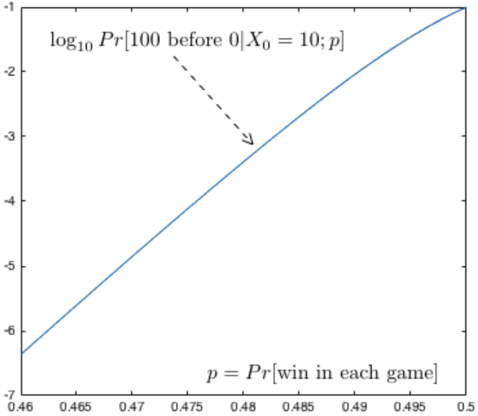

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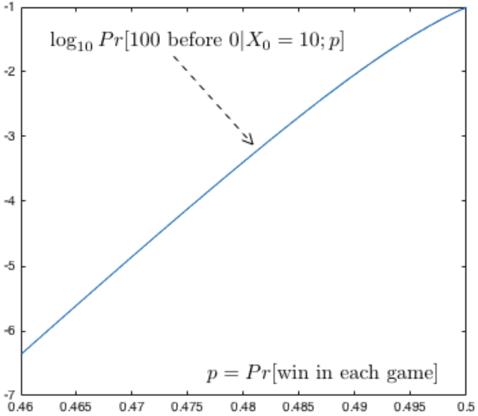

Less than 1 in a 1000.

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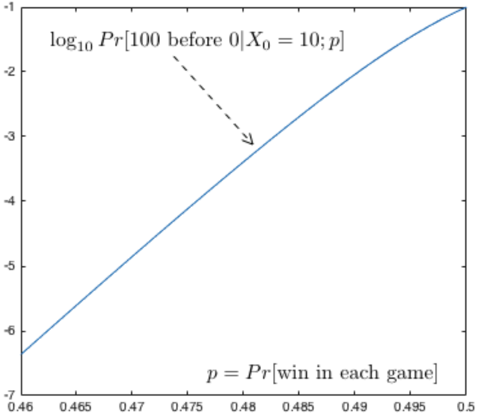

Less than 1 in a 1000. Moral of example:

Here before There - A before B

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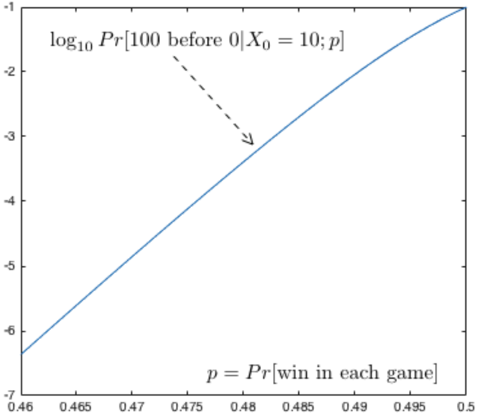

Less than 1 in a 1000. Moral of example: Money in Vegas

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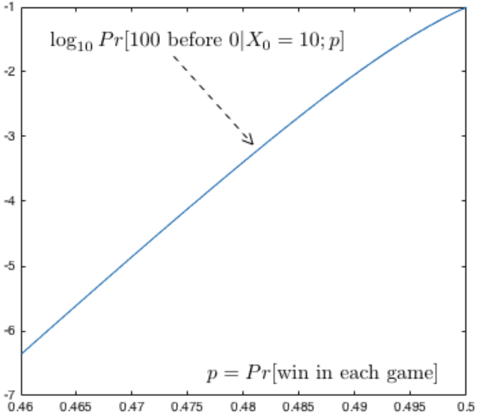

Less than 1 in a 1000. Moral of example: Money in Vegas stays in Vegas.

First Step Equations

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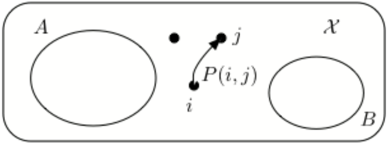

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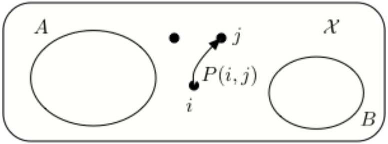

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$$T_A = \min\{n \geq 0 \mid X_n \in A\}$$

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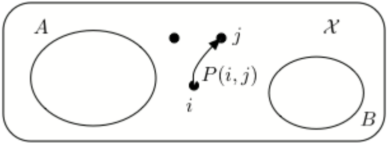

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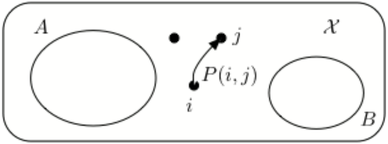

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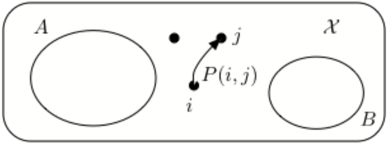

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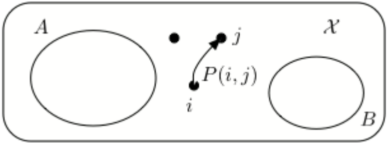

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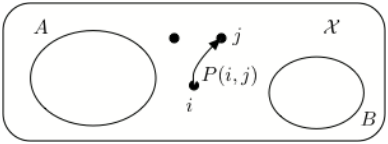

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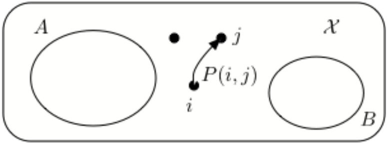

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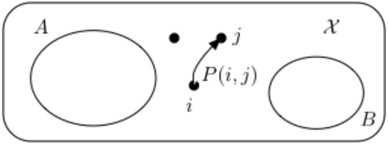

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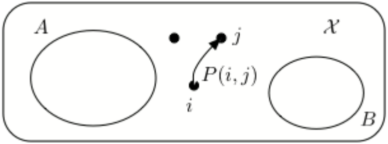

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Flip a fair coin until you get two consecutive Hs.

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What is the expected number of Ts that you see?

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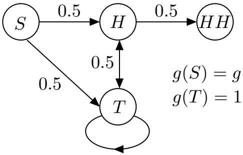

$g(S) = g(H) = g(HH) = 0$
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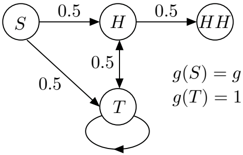

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FSE:

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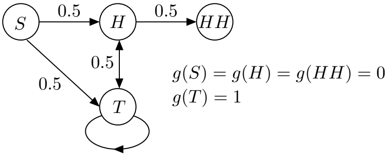

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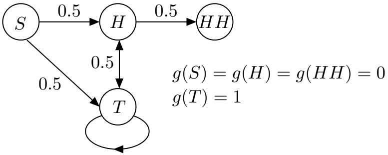

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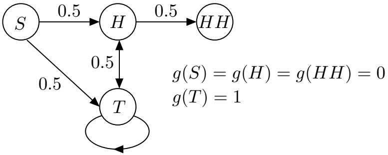

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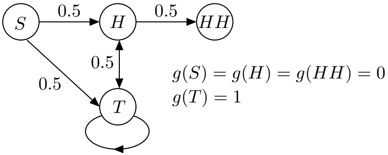

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Solving, we find $\gamma(S) = 2.5$.

Recap

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  - Finite set $\mathscr{X}$; π_0; $P = \{P(i,j), i,j \in \mathscr{X}\}$;
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  - $Pr[X_{n+1} = j \mid X_0, \ldots, X_n = i] = P(i,j), i,j \in \mathscr{X}, n \geq 0.$
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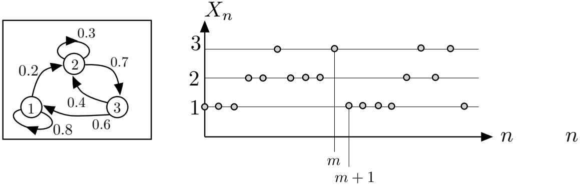
Distribution of X_n



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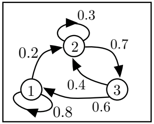

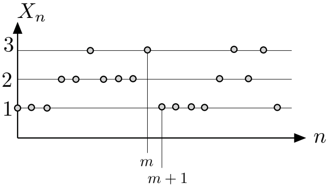

n

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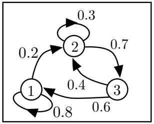

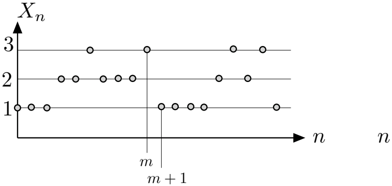

Recall π_n is a distribution over states for X_n.

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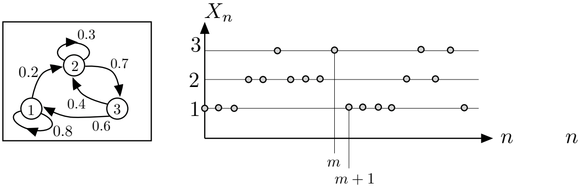

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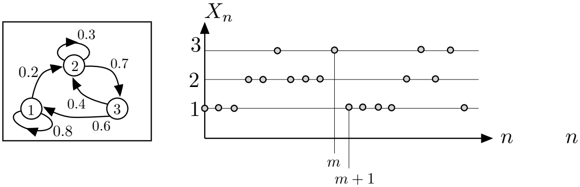

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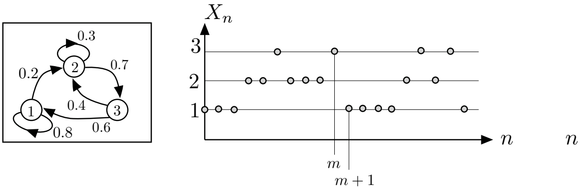

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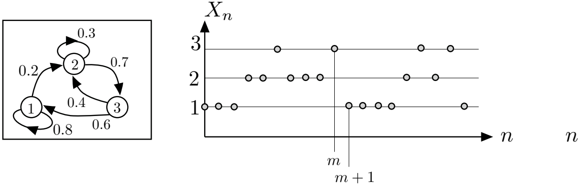

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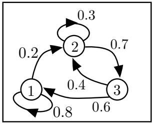

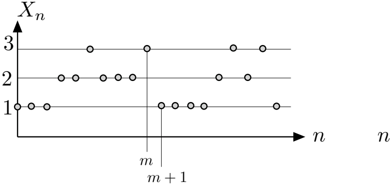

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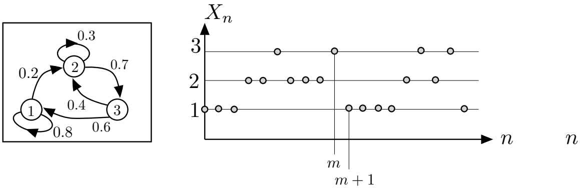

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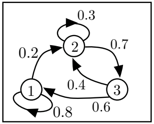

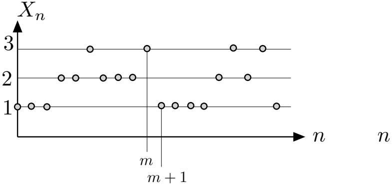

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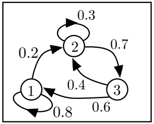

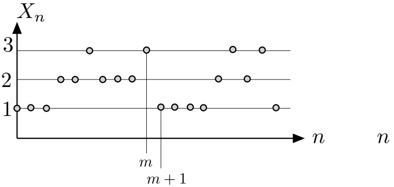

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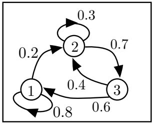

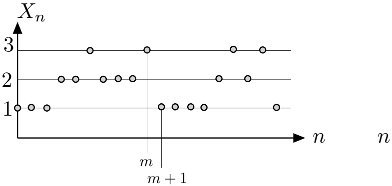

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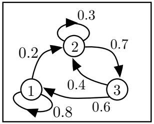

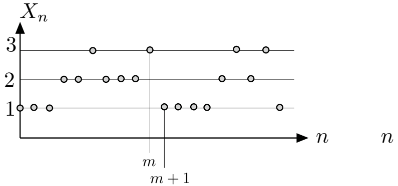

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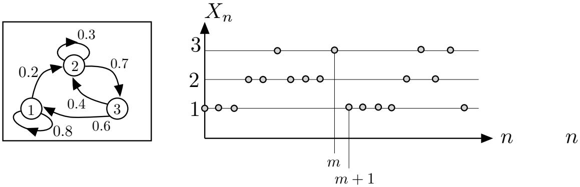

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Sometimes the distribution as $n \to \infty$

Stationary: Example

Example 1:

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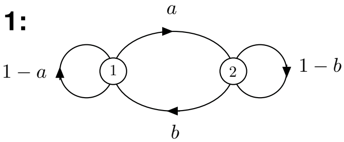

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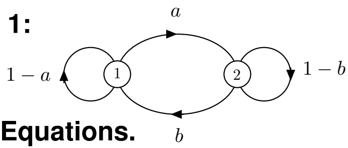

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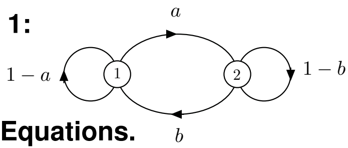

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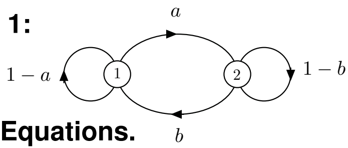

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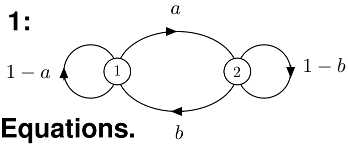

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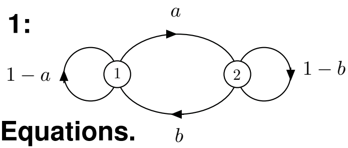

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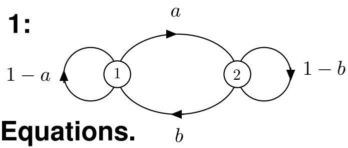

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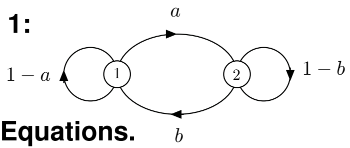

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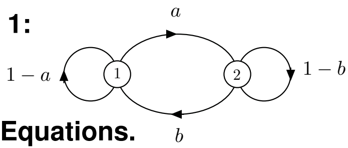

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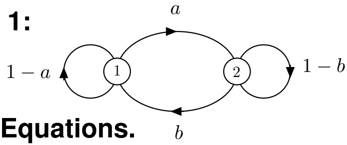

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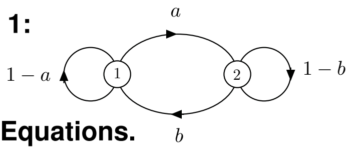

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$$\pi = [\frac{b}{a+b}, \frac{a}{a+b}].$$

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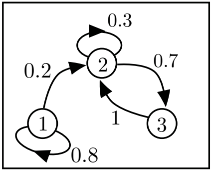

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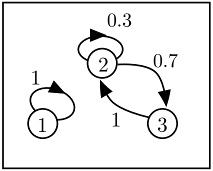

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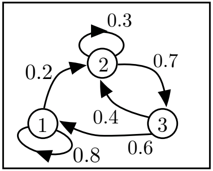

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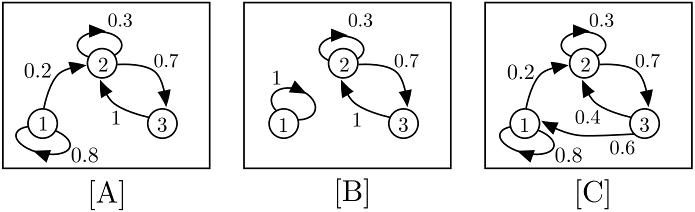

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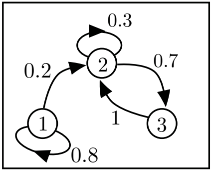

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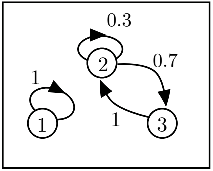

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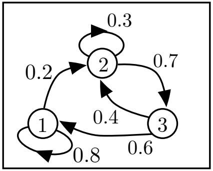

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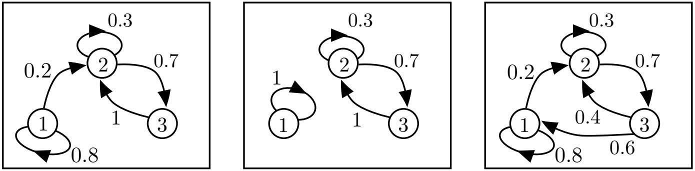

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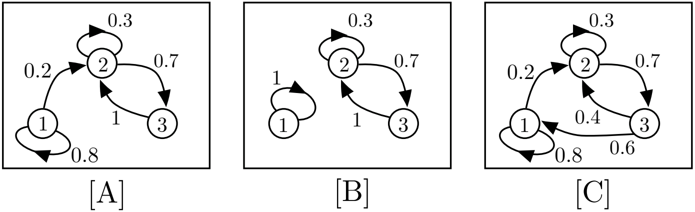

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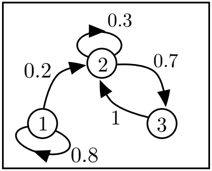

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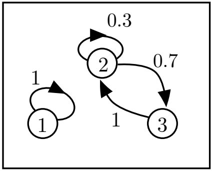

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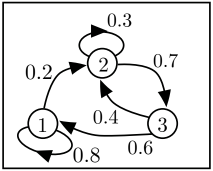

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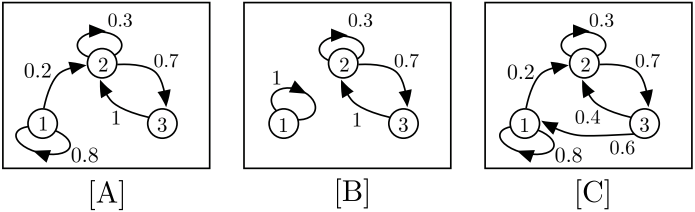

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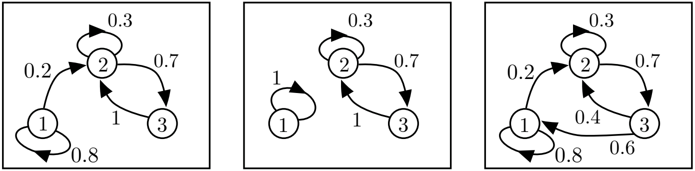

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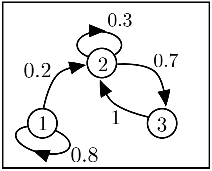

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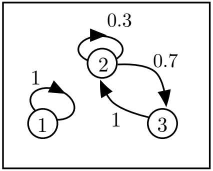

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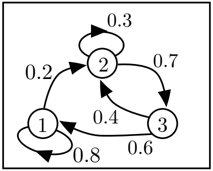

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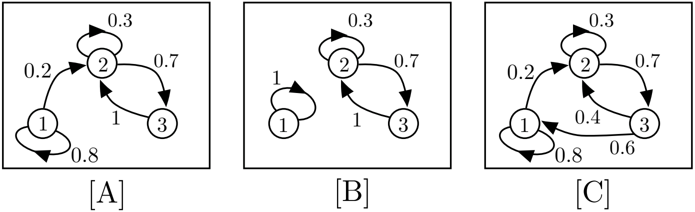

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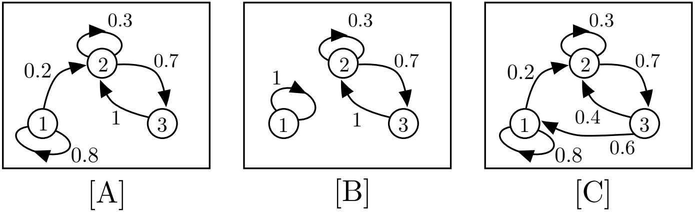

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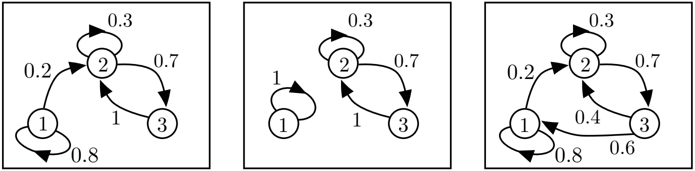

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Only one stationary distribution if irreducible (or connected.)

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Proof: Lecture note 21 gives a plausibility argument.

□

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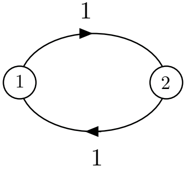

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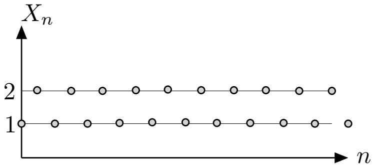

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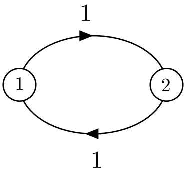

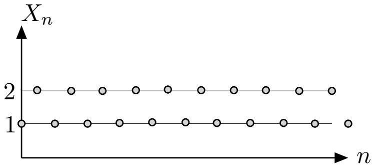

The fraction of time in state 1

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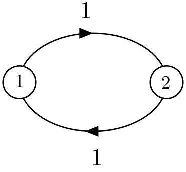

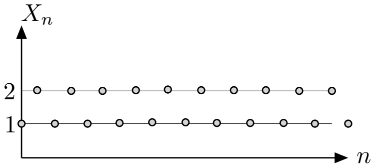

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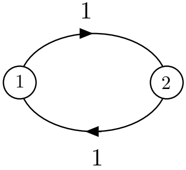

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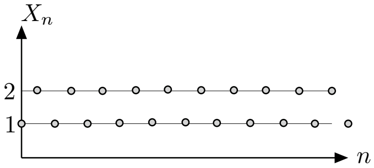

The fraction of time in state 1 converges to $1/2$, which is $\pi(1)$.

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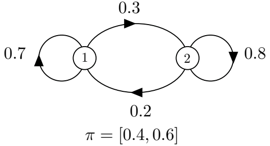

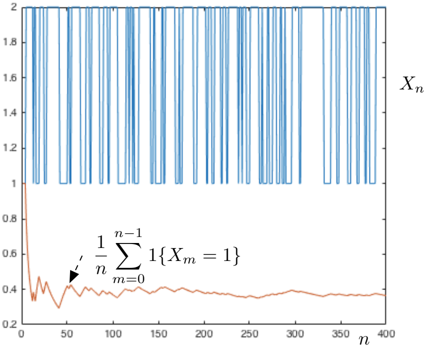

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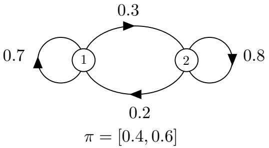

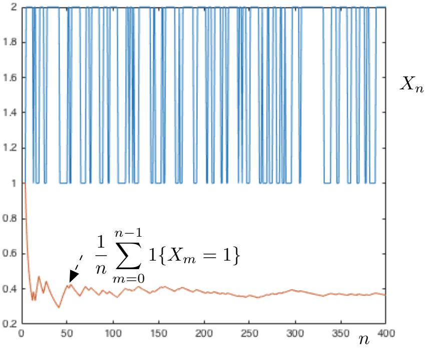

Convergence to Invariant Distribution

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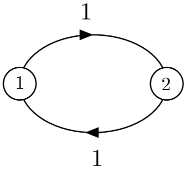

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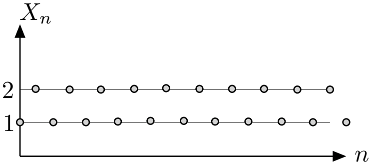

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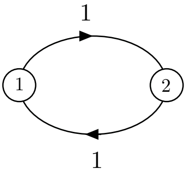

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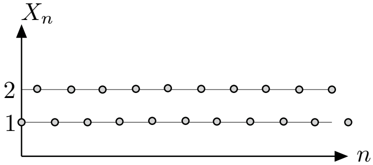

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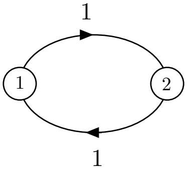

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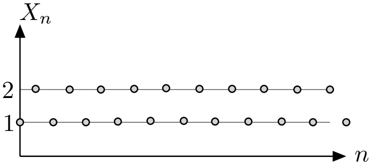

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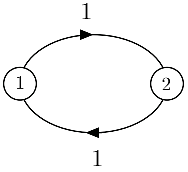

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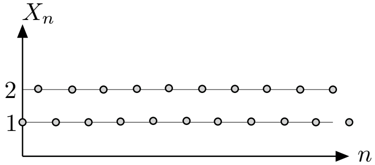

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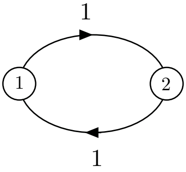

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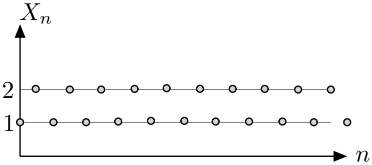

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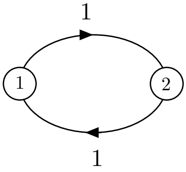

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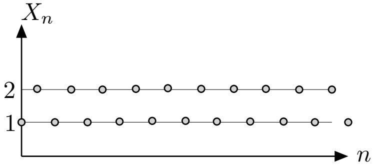

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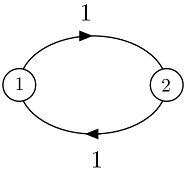

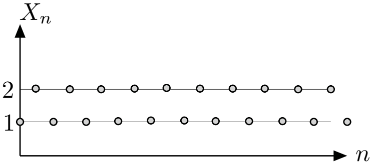

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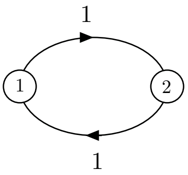

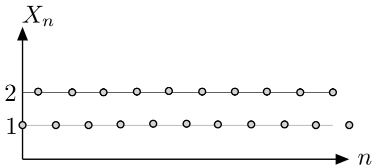

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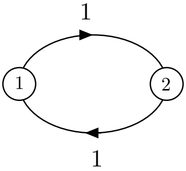

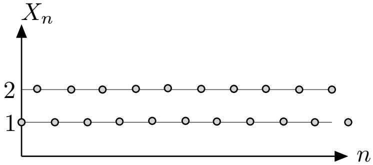

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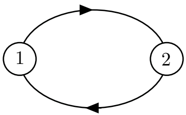

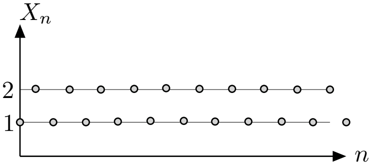

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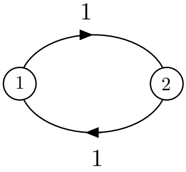

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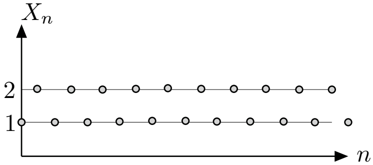

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Notice, all cycles or closed walks have even length.

Convergence to stationary distribution.

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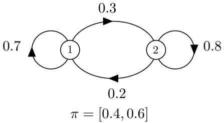

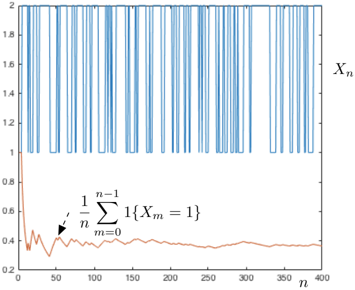

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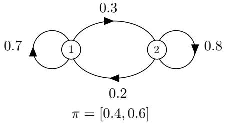

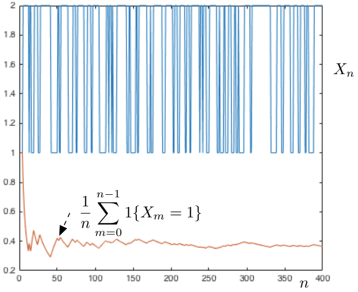

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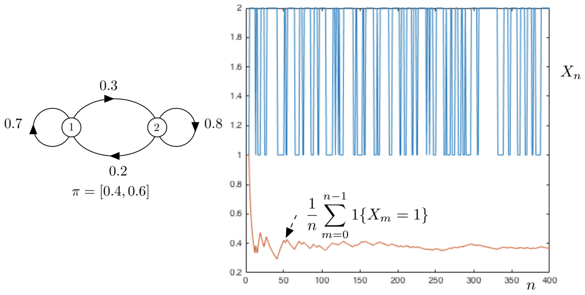

As n gets large the probability of being in either state approaches $1/2$. (The stationary distribution.) Notice cycles of length 1 and 2.

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Definition: Periodicity is gcd of the lengths of all closed walks in irreducible chain.

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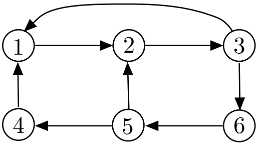
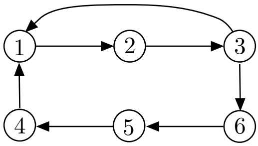


$[A]$



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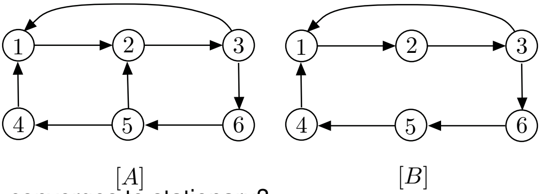

Which one is converges to stationary?

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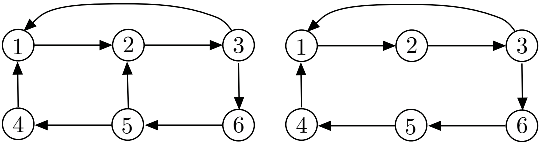

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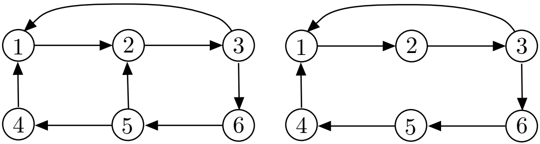

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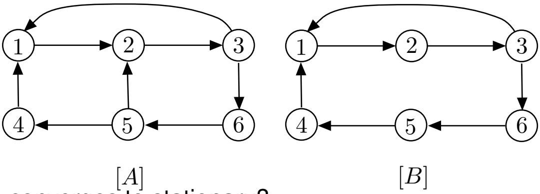

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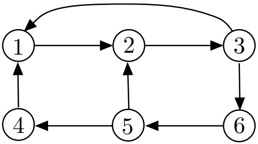

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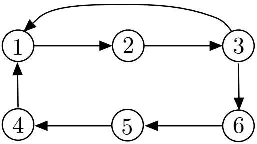

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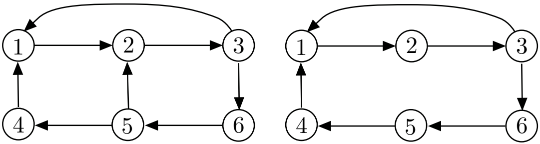

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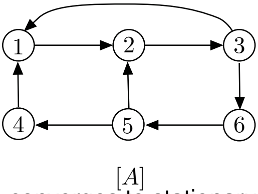

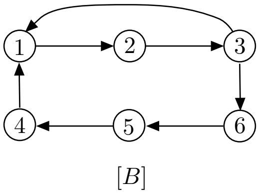

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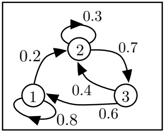

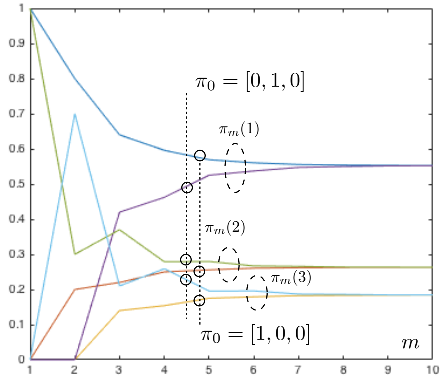

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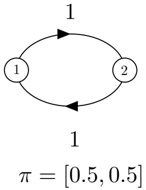

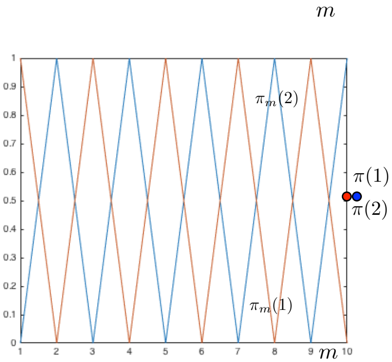

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- FSE: $\beta(i) = 1 + \sum_j P(i,j)\beta(j); \alpha(i) = \sum_j P(i,j)\alpha(j)$.
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- Irreducible + Aperiodic $\Rightarrow \pi_n \to \pi$.
- Calculating π: One finds $\pi = [0, 0 \ldots, 1]Q^{-1}$ where $Q = \cdots$.