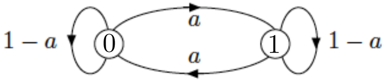
CS70: Markov Chains.

Markov Chains

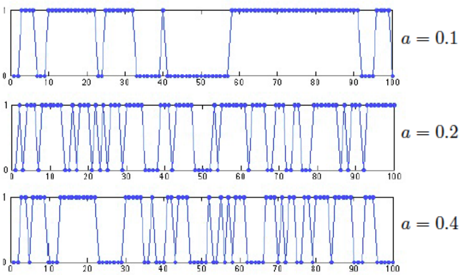
1. Examples
2. Definition
3. Hitting Time.
4. Here before there.
5. Stationary Distribution
6. Peridoicity.

Two-State Markov Chain

Here is a symmetric two-state Markov chain. It describes a random motion in $\{0, 1\}$. Here, *a* is the probability that the state changes in the next step.

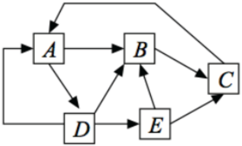


Let's simulate the Markov chain:

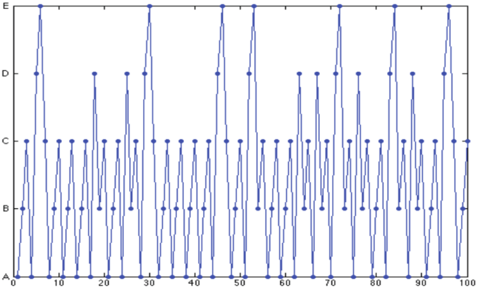


Five-State Markov Chain

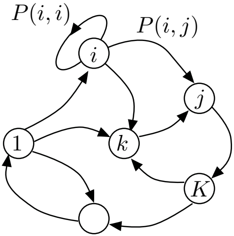
At each step, the MC follows one of the outgoing arrows of the current state, with equal probabilities.



Let's simulate the Markov chain:



Finite Markov Chain: Definition



- A finite set of states: $\mathscr{X} = \{1, 2, \ldots, K\}$
- A probability distribution π_0 on $\mathscr{X}$: $\pi_0(i) \geq 0, \sum_i \pi_0(i) = 1$
- Transition probabilities: $P(i,j)$ for $i,j \in \mathscr{X}$

  $P(i,j) \geq 0, \forall i,j; \sum_j P(i,j) = 1, \forall i$
- $\{X_n, n \geq 0\}$ is defined so that

  $Pr[X_0 = i] = \pi_0(i), i \in \mathscr{X}$ (initial distribution)

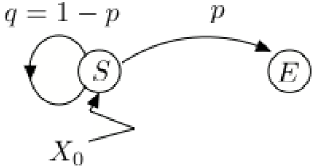
  $Pr[X_{n+1} = j \mid X_0, \ldots, X_n = i] = P(i,j), i,j \in \mathscr{X}$.

Hitting Time - Example 1

Let's flip a coin with $Pr[H] = p$ until we get H. How many flips, on average?

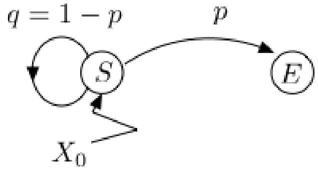
Let's define a Markov chain:

- $X_0 = S$ (start)
- $X_n = S$ for $n \geq 1$, if last flip was T and no H yet
- $X_n = E$ for $n \geq 1$, if we already got H (end)



First Passage Time - Example 1. Poll

Let's flip a coin with $Pr[H] = p$ until we get H. How many flips, on average?



Let $\beta(S)$ be the average time until E, starting from S.

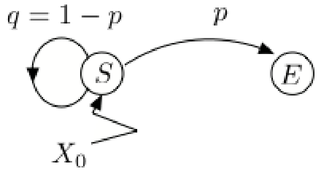
What is correct?

(A) $\beta(S)$ is at least 1.
(B) From S, in one step, go to S with prob. $q = 1 - p$
(C) From S, in one step, go to E with prob. p.
(D) If you go back to S, you are back at S.
(D) $\beta(S) = 1 + q\beta(S) + p0$.

All are correct. (D) is the "Markov property." Only know where you are.

Hitting Time - Example 1

Let's flip a coin with $Pr[H] = p$ until we get H. How many flips, on average?



Let $\beta(S)$ be the average time until E, starting from S.

Then,

$$\beta(S) = 1 + q\beta(S) + p0.$$

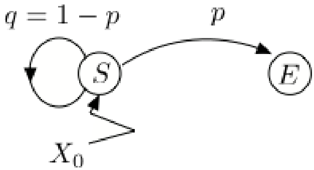
(See next slide.) Hence,

$$\beta(S) = 1 + (1-p)\beta(S) \implies \beta(S) = 1, \text{ so that } \beta(S) = 1/p.$$

Note: Time until E is $G(p)$.
The mean of $G(p)$ is $1/p$!!!

First Passage Time - Example 1

Let's flip a coin with $Pr[H] = p$ until we get H. How many flips, on average?



Let $\beta(S)$ be the average time until E.
Then,

$$\beta(S) = 1 + q\beta(S) + p0.$$

Justification: N – number of steps until E, starting from S.
N' – number of steps until E, after the second visit to S.
And $Z = 1\{\text{first flip } = H\}$. Then,

$$N = 1 + (1 - Z) \times N' + Z \times 0.$$

Z and N' are "independent." $E[N'] = E[N] = \beta(S)$.
Hence, taking expectation,

$$\beta(S) = E[N] = 1 + (1 - p)E[N'] + p0 = 1 + q\beta(S) + p0.$$

Hitting Time - Example 2

Let's flip a coin with $Pr[H] = p$ until we get two consecutive Hs. How many flips, on average?

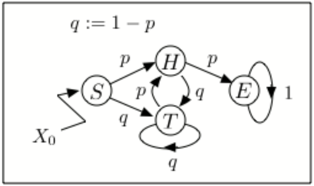
$$H\ T\ H\ T\ T\ T\ H\ T\ H\ T\ H\ T\ T\ H\ T\ H\ H$$

Let's define a Markov chain:

- $X_0 = S$ (start)
- $X_n = E$, if we already got two consecutive Hs (end)
- $X_n = T$, if last flip was T and we are not done
- $X_n = H$, if last flip was H and we are not done

Hitting Time - Example 2

Let's flip a coin with $Pr[H] = p$ until we get two consecutive Hs. How many flips, on average? Here is a picture:



S: Start
H: Last flip $= H$
T: Last flip $= T$
E: Done

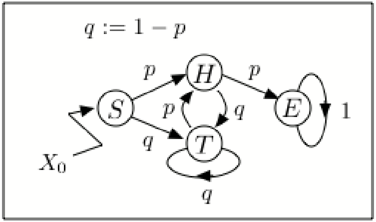
Which one is correct?
(A) $\beta(S) = 1 + p\beta(H) + q\beta(T)$
(B) $\beta(S) = p\beta(H) + q\beta(T)$
(C) $\beta(S) = \beta(S) + q\beta(T) + p\beta(H)$.

(A) Expected time from S to E.

$$\beta(S) = Pr[H]E[\beta(S)|H] + Pr[T]E[\beta(S)|T]$$
$$\beta(S) = p(1 + \beta(H)) + q(1 + \beta(T))$$
$$\beta(S) = 1 + p\beta(H) + q\beta(T)$$

Hitting Time - Example 2

Let's flip a coin with $Pr[H] = p$ until we get two consecutive Hs. How many flips, on average? Here is a picture:



S: Start
H: Last flip $= H$
T: Last flip $= T$
E: Done

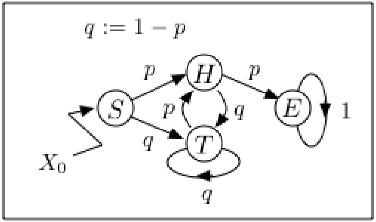
Let $\beta(i)$ be the average time from state i until the MC hits state E.

We claim that (these are called the first step equations)

$$\beta(S) = 1 + p\beta(H) + q\beta(T)$$
$$\beta(H) = 1 + p0 + q\beta(T)$$
$$\beta(T) = 1 + p\beta(H) + q\beta(T).$$

Solving, we find $\beta(S) = 2 + 3qp^{-1} + q^2p^{-2}$. (E.g., $\beta(S) = 6$ if $p = 1/2$.)

Hitting Time - Example 2



S: Start

H: Last flip $= H$

T: Last flip $= T$

E: Done

Let us justify the first step equation for $\beta(T)$. The others are similar.

$N(T)$ – number of steps, starting from T until the MC hits E.
$N(H)$ – be defined similarly.
$N'(T)$ – number of steps after the second visit to T until MC hits E.

$$N(T) = 1 + Z \times N(H) + (1 - Z) \times N'(T)$$

where $Z = 1\{\text{first flip in } T \text{ is } H\}$. Since Z and $N(H)$ are independent, and Z and $N'(T)$ are independent, taking expectations, we get

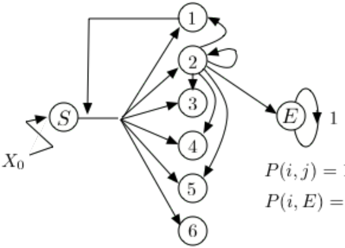
$$E[N(T)] = 1 + pE[N(H)] + qE[N'(T)],$$

i.e.,

$$\beta(T) = 1 + p\beta(H) + q\beta(T).$$

Hitting Time - Example 3

You roll a balanced six-sided die until the sum of the last two rolls is 8. How many times do you have to roll the die, on average?



S = Start; E = Done

i = Last roll is i, not done

$P(S, j) = 1/6, j = 1, \ldots, 6$

$P(1, j) = 1/6, j = 1, \ldots, 6$

$P(i, j) = 1/6, i = 2, \ldots, 6; 8 - i \neq j \in \{1, \ldots, 6\}$

$P(i, E) = 1/6, i = 2, \ldots, 6$

The arrows out of $3, \ldots, 6$ (not shown) are similar to those out of 2.

$$\beta(S) = 1 + \frac{1}{6}\sum_{j=1}^{6}\beta(j); \beta(1) = 1 + \frac{1}{6}\sum_{j=1}^{6}\beta(j); \beta(i) = 1 + \frac{1}{6}\sum_{j=1,\ldots,6; j\neq 8-i}\beta(j), i = 2, \ldots, 6.$$

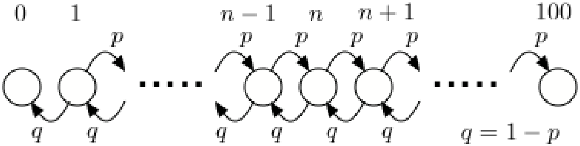
Symmetry: $\beta(2) = \cdots = \beta(6) =: \gamma$. Also, $\beta(1) = \beta(S)$. Thus,

$$\beta(S) = 1 + (5/6)\gamma + \beta(S)/6; \quad \gamma = 1 + (4/6)\gamma + (1/6)\beta(S).$$

$$\Rightarrow \cdots \beta(S) = 8.4.$$

Here before There - A before B

Game of “heads or tails” using coin with ‘heads’ probability $p < 0.5$.
Start with \$10.
Each step, flip yields ‘heads’, earn \$1. Otherwise, lose \$1.
What is the probability that you reach \$100 before \$0?

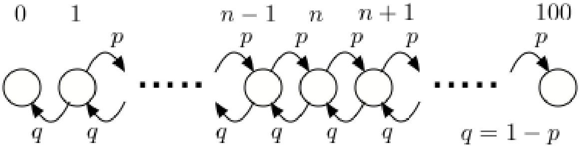


Let $\alpha(n)$ be the probability of reaching 100 before 0, starting from n, for $n = 0, 1, \ldots, 100$.
Which equations are correct?
(A) $\alpha(0) = 0$
(B) $\alpha(0) = 1$.
(C) $\alpha(100) = 1$.
(D) $\alpha(n) = 1 + p\alpha(n+1) + q\alpha(n-1), 0 < n < 100$.
(E) $\alpha(n) = p\alpha(n+1) + q\alpha(n-1), 0 < n < 100$.

(B) is incorrect, 0 is bad.
(D) is incorrect. Confuses expected hitting time with A before B.

Here before There - A before B

Game of "heads or tails" using coin with 'heads' probability $p < 0.5$.
Start with \$10.
Each step, flip yields 'heads', earn \$1. Otherwise, lose \$1.
What is the probability that you reach \$100 before \$0?



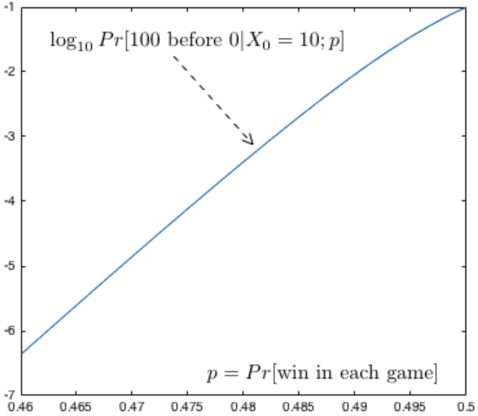
Let $\alpha(n)$ be the probability of reaching 100 before 0, starting from n, for $n = 0, 1, \ldots, 100$.

$$\alpha(0) = 0; \alpha(100) = 1.$$
$$\alpha(n) = p\alpha(n+1) + q\alpha(n-1), 0 < n < 100.$$

$$\Rightarrow \alpha(n) = \frac{1-\rho^n}{1-\rho^{100}} \text{ with } \rho = qp^{-1}. \text{ (See LN 22)}$$

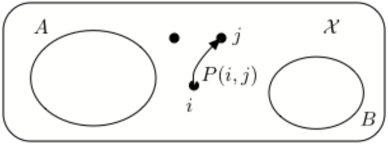
Here before There - A before B

Game of "heads or tails" using coin with 'heads' probability $p = .48$.
Start with \$10.
Each step, flip yields 'heads', earn \$1. Otherwise, lose \$1.
What is the probability that you reach \$100 before \$0?



Less than 1 in a 1000. Moral of example: Money in Vegas stays in Vegas.

First Step Equations



Let X_n be a MC on $\mathscr{X}$ and $A, B \subset \mathscr{X}$ with $A \cap B = \emptyset$. Define

$$T_A = \min\{n \geq 0 \mid X_n \in A\} \text{ and } T_B = \min\{n \geq 0 \mid X_n \in B\}.$$

For $\beta(i) = E[T_A \mid X_0 = i]$, first step equations are:

$$\beta(i) = 0, i \in A$$
$$\beta(i) = 1 + \sum_j P(i,j)\beta(j), i \notin A$$

For $\alpha(i) = Pr[T_A < T_B \mid X_0 = i], i \in \mathscr{X}$,, first step equations are:

$$\alpha(i) = 1, i \in A$$
$$\alpha(i) = 0, i \in B$$
$$\alpha(i) = \sum_j P(i,j)\alpha(j), i \notin A \cup B.$$

Accumulating Rewards

Let X_n be a Markov chain on $\mathscr{X}$ with P. Let $A \subset \mathscr{X}$

Let also $g : \mathscr{X} \to \Re$ be some function.

Define

$$\gamma(i) = E[\sum_{n=0}^{T_A} g(X_n) | X_0 = i], i \in \mathscr{X}.$$

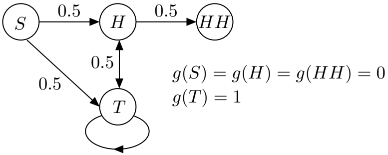
Then

$$\gamma(i) = \begin{cases} g(i), & \text{if } i \in A \\ g(i) + \sum_j P(i,j)\gamma(j), & \text{otherwise.} \end{cases}$$

Example

Flip a fair coin until you get two consecutive Hs.

What is the expected number of Ts that you see?



FSE:

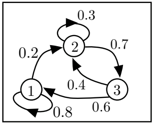
$$\gamma(S) = 0 + 0.5\gamma(H) + 0.5\gamma(T)$$
$$\gamma(H) = 0 + 0.5\gamma(HH) + 0.5\gamma(T)$$
$$\gamma(T) = 1 + 0.5\gamma(H) + 0.5\gamma(T)$$
$$\gamma(HH) = 0.$$

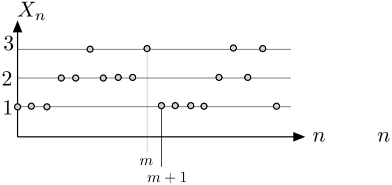
Solving, we find $\gamma(S) = 2.5$.

Recap

- Markov Chain:
  - Finite set $\mathscr{X}$; π_0; $P = \{P(i,j), i,j \in \mathscr{X}\}$;
  - $Pr[X_0 = i] = \pi_0(i), i \in \mathscr{X}$
  - $Pr[X_{n+1} = j \mid X_0, \ldots, X_n = i] = P(i,j), i,j \in \mathscr{X}, n \geq 0.$
  - Note:
    $Pr[X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n] = \pi_0(i_0)P(i_0,i_1)\cdots P(i_{n-1},i_n).$
- First Passage Time:
  - $A \cap B = \emptyset; \beta(i) = E[T_A|X_0 = i]; \alpha(i) = P[T_A < T_B|X_0 = i]$
  - $\beta(i) = 1 + \sum_j P(i,j)\beta(j);$
  - $\alpha(i) = \sum_j P(i,j)\alpha(j).\ \alpha(A) = 1, \alpha(B) = 0.$

Distribution of X_n



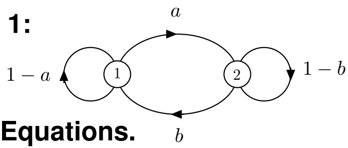


Recall π_n is a distribution over states for X_n.

Stationary distribution: $\pi = \pi P$.
  Distribution over states is the same before/after transition.
    probability entering i: $\sum_{i,j} P(j,i)\pi(j)$.
    probability leaving i: π_i.
    are Equal!
Distribution same after one step.
Questions? Does one exist? Is it unique?
If it exists and is unique. Then what?
  Sometimes the distribution as $n \to \infty$

Stationary: Example

Example 1:



$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

Balance Equations.

$$\begin{aligned} \pi P = \pi \quad &\Leftrightarrow \quad [\pi(1), \pi(2)] \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} = [\pi(1), \pi(2)] \\ &\Leftrightarrow \quad \pi(1)(1-a) + \pi(2)b = \pi(1) \text{ and } \pi(1)a + \pi(2)(1-b) = \pi(2) \\ &\Leftrightarrow \quad \pi(1)a = \pi(2)b. \end{aligned}$$

These equations are redundant! We have to add an equation: $\pi(1) + \pi(2) = 1$. Then we find

$$\pi = [\frac{b}{a+b}, \frac{a}{a+b}].$$

Stationary distributions: Example 2

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\pi P = \pi \Leftrightarrow [\pi(1), \pi(2)] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [\pi(1), \pi(2)] \Leftrightarrow \pi(1) = \pi(1) \text{ and } \pi(2) = \pi(2).$$

Every distribution is invariant for this Markov chain. Since $X_n = X_0$ for all n. Hence, $Pr[X_n = i] = Pr[X_0 = i], \forall (i, n)$.

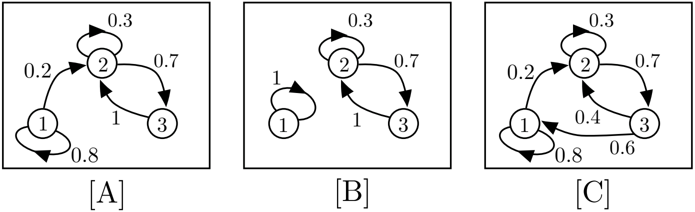
Discussion.
We have seen a chain with one stationary,
and a chain with many.

When is here just one?

Irreducibility.

Definition A Markov chain is irreducible if it can go from every state i to every state j (possibly in multiple steps).

Examples:



[A] is not irreducible. It cannot go from (2) to (1).

[B] is not irreducible. It cannot go from (2) to (1).

[C] is irreducible. It can go from every i to every j.

If you consider the graph with arrows when $P(i,j) > 0$, irreducible means that there is a single connected component.

Existence and uniqueness of Invariant Distribution

Theorem A finite irreducible Markov chain has one and only one invariant distribution.

That is, there is a unique positive vector $\pi = [\pi(1), \ldots, \pi(K)]$ such that $\pi P = \pi$ and $\sum_k \pi(k) = 1$.

Ok. Now.
Only one stationary distribution if irreducible (or connected.)

Long Term Fraction of Time in States

Theorem Let X_n be an irreducible Markov chain with invariant distribution π.

Then, for all i,

$$\frac{1}{n}\sum_{m=0}^{n-1} 1\{X_m = i\} \rightarrow \pi(i), \text{ as } n \rightarrow \infty.$$

The left-hand side is the fraction of time that $X_m = i$ during steps $0, 1, \ldots, n-1$. Thus, this fraction of time approaches $\pi(i)$.

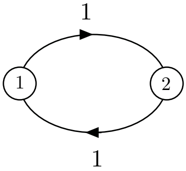
Proof: Lecture note 21 gives a plausibility argument.

$\square$

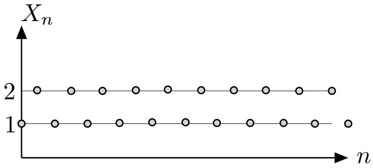
Long Term Fraction of Time in States

Theorem Let X_n be an irreducible Markov chain with invariant distribution π. Then, for all i, $\frac{1}{n}\sum_{m=0}^{n-1} 1\{X_m = i\} \to \pi(i)$, as $n \to \infty$.

Example 1:



$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \pi P = \pi \Rightarrow \pi = [1/2, 1/2]$$

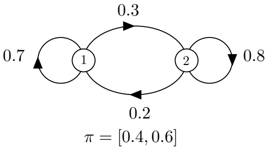


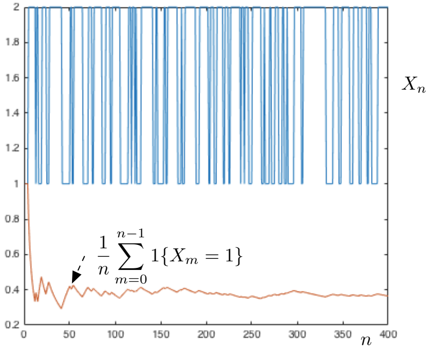
The fraction of time in state 1 converges to $1/2$, which is $\pi(1)$.

Long Term Fraction of Time in States

Theorem Let X_n be an irreducible Markov chain with invariant distribution π. Then, for all i, $\frac{1}{n}\sum_{m=0}^{n-1} 1\{X_m = i\} \to \pi(i)$, as $n \to \infty$.

Example 2:

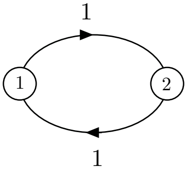




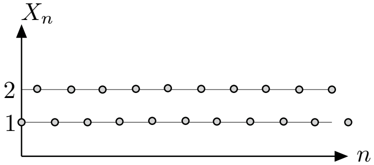
Convergence to Invariant Distribution

Question: Assume that the MC is irreducible. Does π_n approach the unique invariant distribution π?

Answer: Not necessarily. Here is an example:



$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \pi P = \pi \Rightarrow \pi = [1/2, 1/2]$$



Assume $X_0 = 1$. Then $X_1 = 2, X_2 = 1, X_3 = 2, \ldots$.

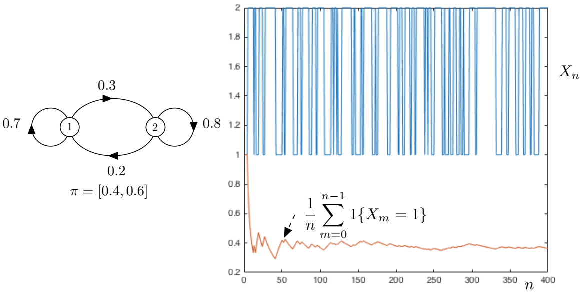
Thus, if $\pi_0 = [1, 0]$, $\pi_1 = [0, 1], \pi_2 = [1, 0], \pi_3 = [0, 1]$, etc.

Hence, π_n does not converge to $\pi = [1/2, 1/2]$.
Notice, all cycles or closed walks have even length.

Convergence to stationary distribution.

Theorem Let X_n be an irreducible Markov chain with invariant distribution π. Then, for all i, $\frac{1}{n}\sum_{m=0}^{n-1} 1\{X_m = i\} \to \pi(i)$, as $n \to \infty$.

Example 2:



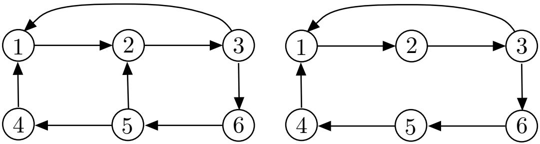
As n gets large the probability of being in either state approaches $1/2$. (The stationary distribution.) Notice cycles of length 1 and 2.

Periodicity

Definition: Periodicity is gcd of the lengths of all closed walks in irreducible chain.   Previous example: 2.

Definition If periodicity is 1, Markov chain is said to be aperiodic. Otherwise, it is periodic.

Example



$[A]$ $[B]$

Which one is converges to stationary?

(A) [A]

(B) [B]

(C) both

(D) neither.

(A).

[A]: Closed walks of length 3 and length 4 $\implies$ periodicity = 1.

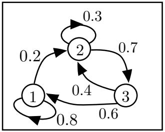
[B]: All closed walks multiple of 3 $\implies$ periodicity =2.

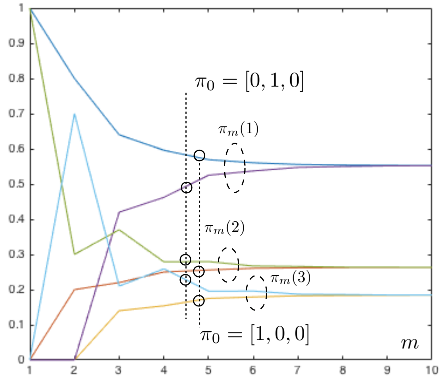
Convergence of π_n

Theorem Let X_n be an irreducible and aperiodic Markov chain with invariant distribution π. Then, for all $i \in \mathscr{X}$,

$$\pi_n(i) \to \pi(i), \text{ as } n \to \infty.$$

Example





Convergence of π_n

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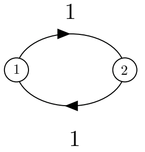
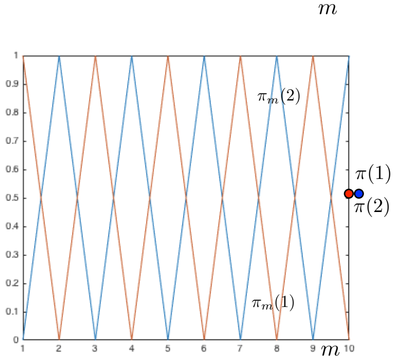
Example



$\pi = [0.5, 0.5]$



Summary

$\boxed{\text{Markov Chains}}$

- Markov Chain: $Pr[X_{n+1} = j | X_0, \ldots, X_n = i] = P(i,j)$
- FSE: $\beta(i) = 1 + \sum_j P(i,j)\beta(j); \alpha(i) = \sum_j P(i,j)\alpha(j)$.
- $\pi_n = \pi_0 P^n$
- π is invariant iff $\pi P = \pi$
- Irreducible $\Rightarrow$ one and only one invariant distribution π
- Irreducible $\Rightarrow$ fraction of time in state i approaches $\pi(i)$
- Irreducible + Aperiodic $\Rightarrow \pi_n \to \pi$.
- Calculating π: One finds $\pi = [0, 0 \ldots, 1]Q^{-1}$ where $Q = \cdots$.