Outline

Linear Regression: wrapup.

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Poisson Distribution: Sum of two Poissons is Poisson.

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Dividing by $var(Y)$, one gets reduction: $\frac{(cov(X,Y))^2}{var(Y)var(Y)} = (corr(X,Y))^2 = r^2$.

LR: Another Figure

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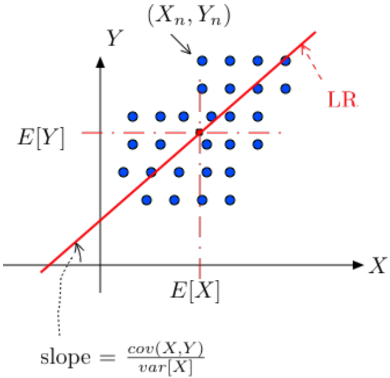

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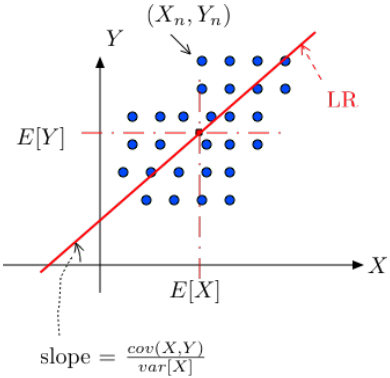

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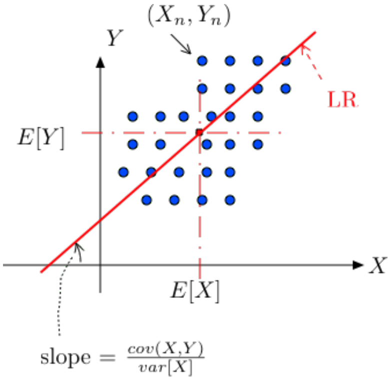

Note that

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Continuous compounded interest: rate r.

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for large n.

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Continuous compounded interest: rate r.
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How do I love e?

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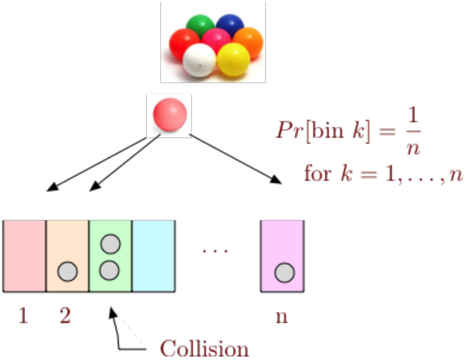

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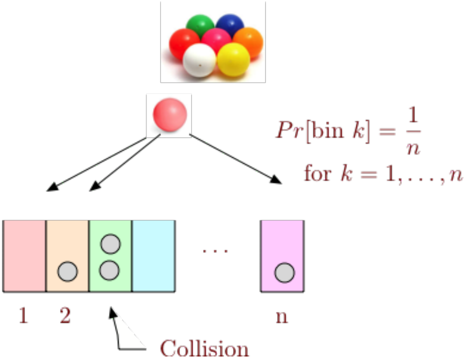

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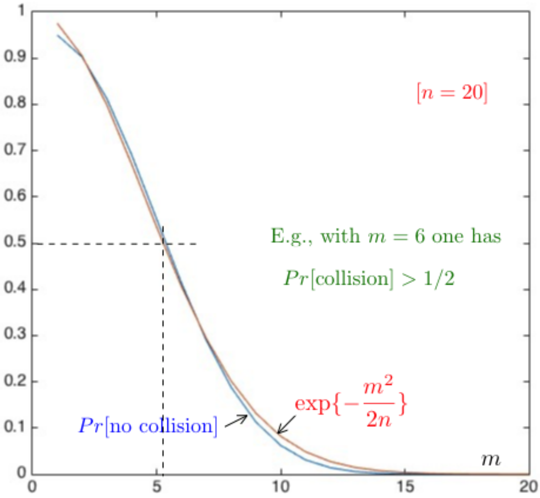

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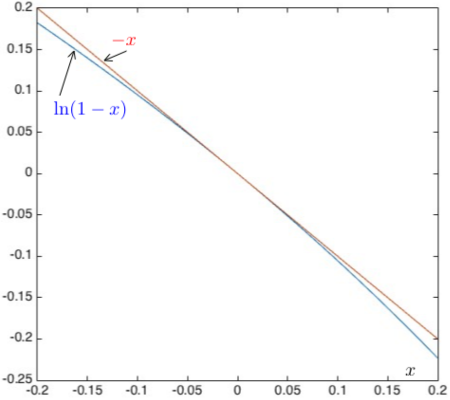
(†) $1 + 2 + \cdots + m - 1 = (m-1)m/2$.

Approximation



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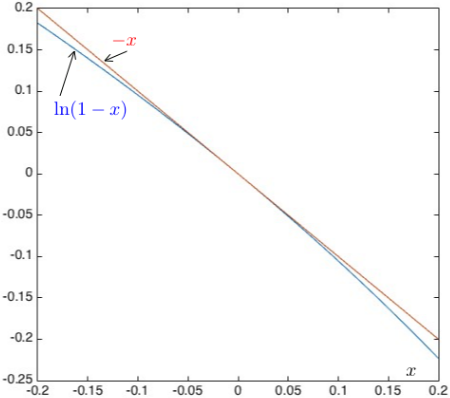

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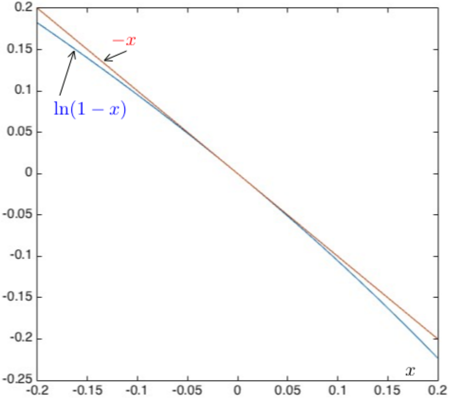

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Hence, $-x \approx \ln(1-x)$ for $|x| \ll 1$.

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& Pr[\text{no collision}] \approx 1 - 10^{-3} \Leftrightarrow m^2/(2n) \approx 10^{-3} \\
& \Leftrightarrow 2n \approx m^2 10^3 \Leftrightarrow 2^{b+1} \approx m^2 2^{10} \\
& \Leftrightarrow b + 1 \approx 10 + 2\log_2(m) \approx 10 + 2.9\ln(m).
\end{aligned}
$$

Checksums!

Consider a set of m files.
Each file has a checksum of b bits.
How large should b be for $Pr[\text{share a checksum}] \leq 10^{-3}$?

Claim: $b \geq 2.9\ln(m)+9$.

Proof:

Let $n = 2^b$ be the number of checksums.
We know $Pr[\text{no collision}] \approx \exp\{-m^2/(2n)\} \approx 1 - m^2/(2n)$. Hence,

$$
\begin{aligned}
Pr[\text{no collision}] &\approx 1 - 10^{-3} \Leftrightarrow m^2/(2n) \approx 10^{-3} \\
&\Leftrightarrow 2n \approx m^2 10^3 \Leftrightarrow 2^{b+1} \approx m^2 2^{10} \\
&\Leftrightarrow b+1 \approx 10 + 2\log_2(m) \approx 10 + 2.9\ln(m).
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Note: $\log_2(x) = \log_2(e)\ln(x) \approx 1.44\ln(x)$.

Coupon Collector Problem.

There are n different baseball cards.
(Brian Wilson, Jackie Robinson, Roger Hornsby, ...)

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(a) $Pr[\text{miss one specific item}] \approx e^{-\frac{m}{n}}$

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One random baseball card in each cereal box.

Theorem: If you buy m boxes,

(a) $Pr[\text{miss one specific item}] \approx e^{-\frac{m}{n}}$

(b) $Pr[\text{miss any one of the items}] \leq ne^{-\frac{m}{n}}$.

Coupon Collector Problem: Analysis.

Event A_m = ‘fail to get Brian Wilson in m cereal boxes’

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For $p_m = \frac{1}{2}$, we need around $n \ln 2 \approx 0.69n$ boxes.

Collect all cards?

Experiment: Choose m cards at random with replacement.

Collect all cards?

Experiment: Choose *m* cards at random with replacement.

Events: E_k = 'fail to get player k' , for k = 1, . . . , n

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Probability of failing to get at least one of these *n* players:

$$p := Pr[E_1 \cup E_2 \cdots \cup E_n]$$

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$$p = Pr[E_1 \cup E_2 \cdots \cup E_n] \leq Pr[E_1] + Pr[E_2] \cdots Pr[E_n].$$

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$$Pr[E_k] \approx e^{-\frac{m}{n}}, k = 1, \ldots, n.$$

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Plug in and get

$$p \leq ne^{-\frac{m}{n}}.$$

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E.g., $n = 10^2 \Rightarrow m = 530$;

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E.g., $n = 10^2 \Rightarrow m = 530; n = 10^3 \Rightarrow m = 7600.$

Time to collect coupons

X-time to get n coupons.

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X_1 - time to get first coupon.

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Pr[“get second coupon”|“got milk ”]

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Pr[“get second coupon”|“got ~~milk~~ first coupon”] $= \frac{n-1}{n}$

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$$\begin{aligned} E[X] &= E[X_1] + \cdots + E[X_n] = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1} \\ &= n(1 + \frac{1}{2} + \cdots + \frac{1}{n}) =: nH(n) \end{aligned}$$

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Review: Harmonic sum

$$H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \int_1^n \frac{1}{x} dx = \ln(n).$$

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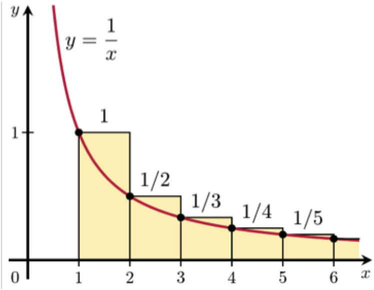

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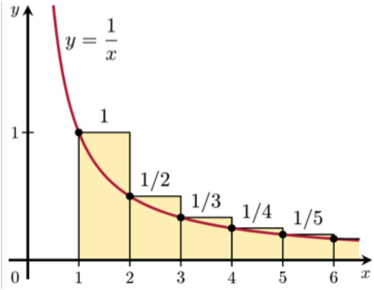

A good approximation is

$H(n) \approx \ln(n) + \gamma$ where $\gamma \approx 0.58$ (Euler-Mascheroni constant).

Simplest..

Load balance: m balls in n bins.

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For simplicity: n balls in n bins.

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Load balance: m balls in n bins.

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Round robin:

Simplest..

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Centralized!

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Load balance: m balls in n bins.

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Details: both could arrive with probability $\lambda\mu/n^2$.
But this goes to zero as $n \to \infty$.
(Like λ^2/n^2 in previous derivation)

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Probability Space: Ω, $Pr : \Omega \rightarrow [0,1]$, $\sum_{\omega \in \Omega} Pr(w) = 1$.

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$Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$

Simple Total Probability: $Pr[B] = Pr[A \cap B] + Pr[\overline{A} \cap B]$.

Conditional Probability: $Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}$.

Simple Product Rule: $Pr[A \cap B] = Pr[A|B] Pr[B]$.

Bayes Rule: $Pr[A|B] = \frac{Pr[B|A] Pr[B]}{Pr[B]}$

Inference:
Have one of two coins. Flip coin, which coin do you have?
Got positive test result. What is probability you have disease?

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Note: Probability Mass Function $\equiv$ Distribution.

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For X_i with $Var(X_i) = \sigma^2$.
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