Today

Estimation.

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MMSE: Best Function that predicts X from Y.

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Applications to random processes.

Estimation: Expectation and Mean Squared Error.

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Definitions Let X and Y be RVs on Ω.

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Proof: $E[Y|X=x] = E[Y|A]$ with $A = \{\omega : X(\omega) = x\}$. $\square$

Properties of CE

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(e) Let $h(X)=1$ in (d).

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Note that (d) says that

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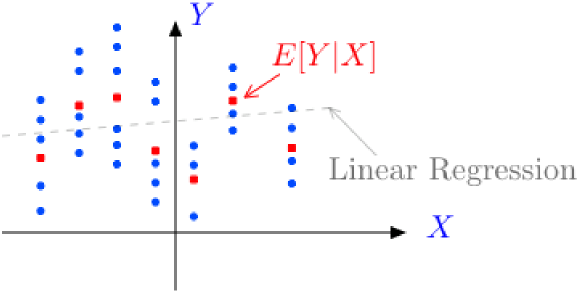

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Thus, $E[(Y - h(X))^2] \geq E[(Y - g(X))^2]$. $\square$

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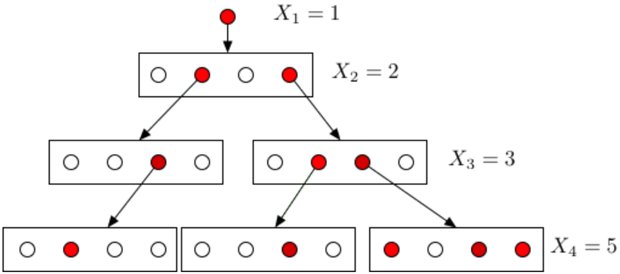

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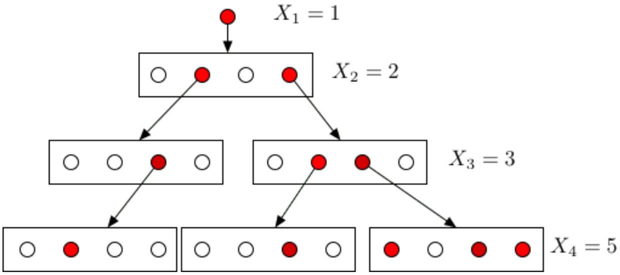
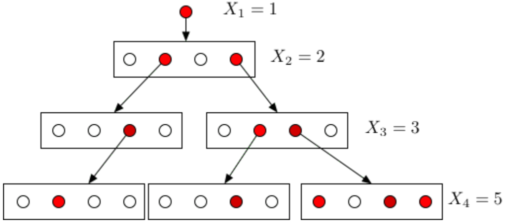


In this example, $d = 4$.

Application: Going Viral



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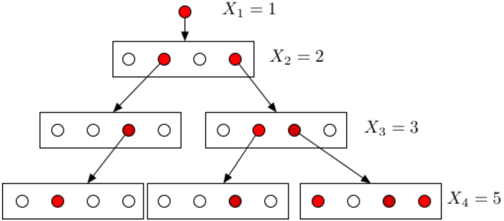
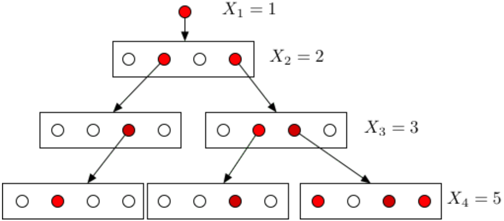


Fact:

Application: Going Viral



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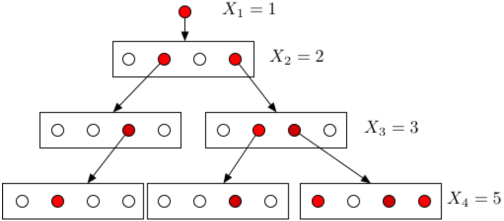
Application: Going Viral



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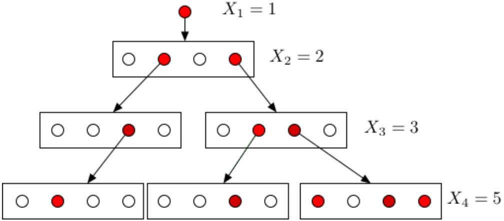
Application: Going Viral



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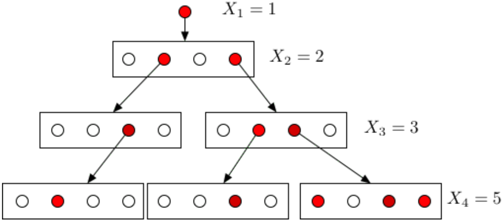

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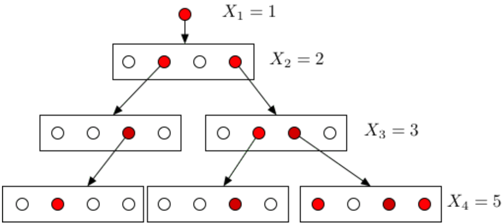

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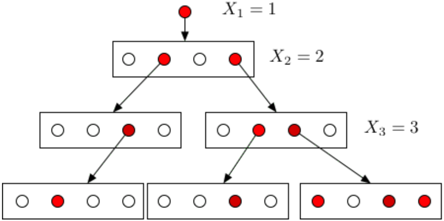

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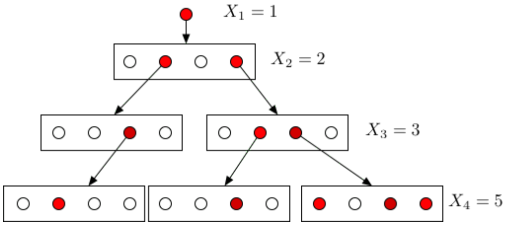

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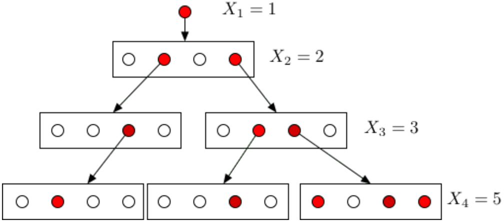

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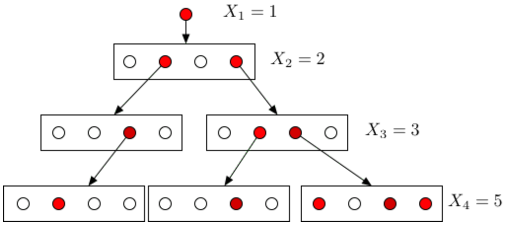

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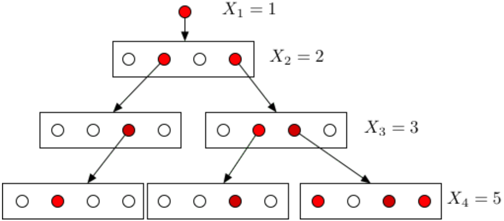

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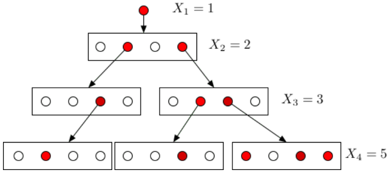

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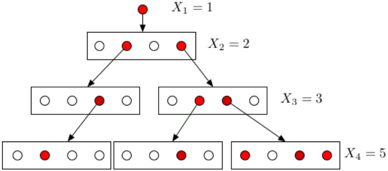

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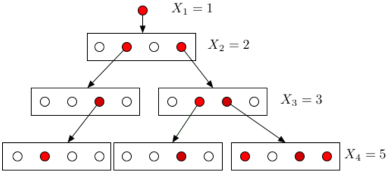

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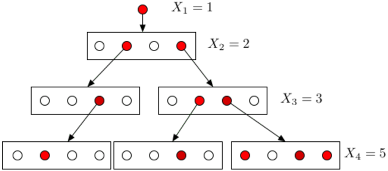

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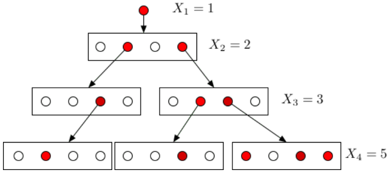

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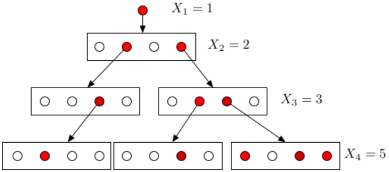

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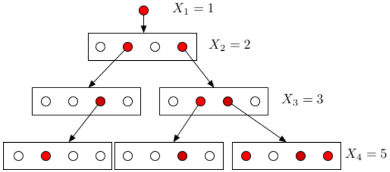

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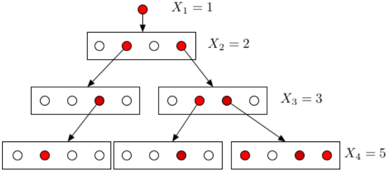

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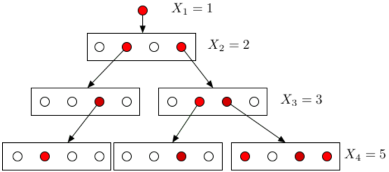

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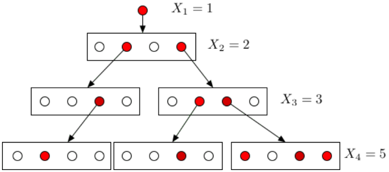

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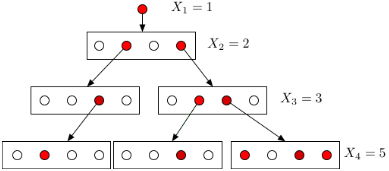

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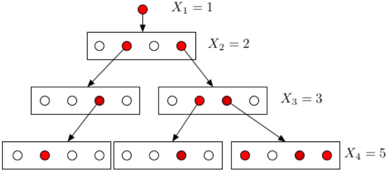

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We conclude as before.

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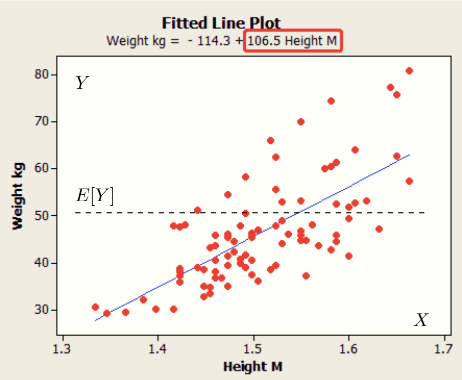

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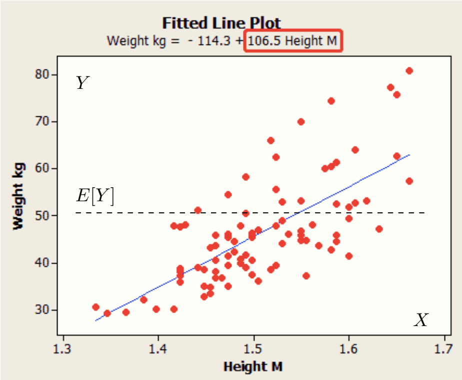

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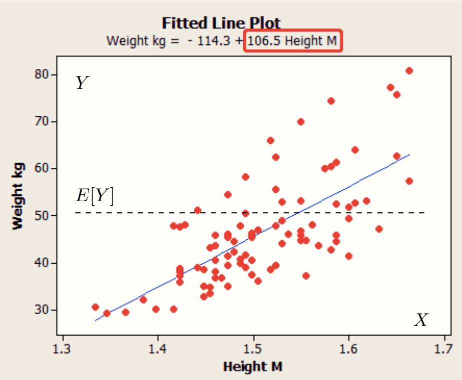

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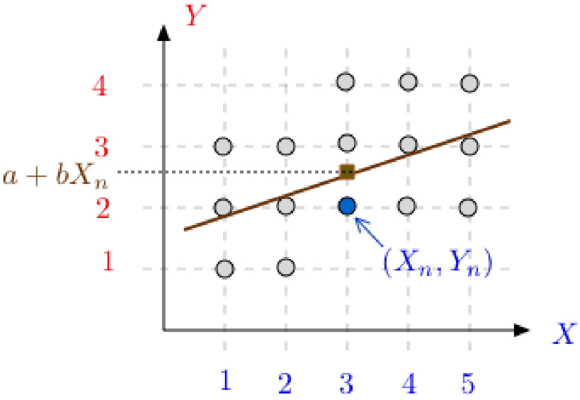

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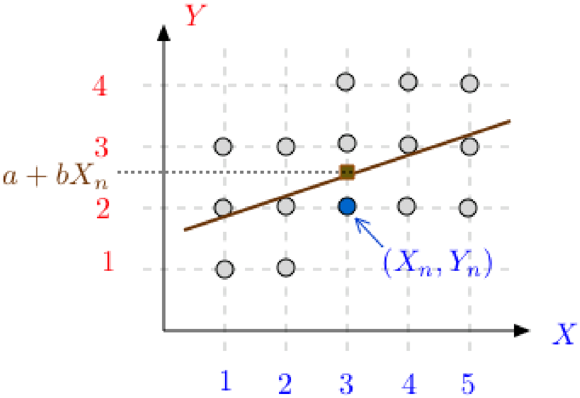

The line $Y = a + bX$ is the linear regression.

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This shows that $E[(Y - \hat{Y})^2] \leq E[(Y - a - bX)^2]$, for all (a, b).

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LLSE

$LLSE[Y|X]$ - best guess for Y given X.

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LLSE

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Estimation Error

We saw that the LLSE of Y given X is

$$L[Y|X] = \hat{Y} = E[Y] + \frac{cov(X,Y)}{var(X)}(X - E[X]).$$

How good is this estimator?
Or what is the mean squared estimation error?

We find

$$E[|Y - L[Y|X]|^2] = E[(Y - E[Y] - (cov(X,Y)/var(X))(X - E[X]))^2]$$

$$= E[(Y - E[Y])^2] - 2\frac{cov(X,Y)}{var(X)}E[(Y - E[Y])(X - E[X])]$$

$$+ (\frac{cov(X,Y)}{var(X)})^2 E[(X - E[X])^2]$$

$$= var(Y) - \frac{cov(X,Y)^2}{var(X)}.$$

Without observations, the estimate is $E[Y]$. The error is $var(Y)$. Observing X reduces the error.

Estimation Error: A Picture

We saw that

$$L[Y|X] = \hat{Y} = E[Y] + \frac{cov(X,Y)}{var(X)}(X - E[X])$$

Estimation Error: A Picture

We saw that

$$L[Y|X] = \hat{Y} = E[Y] + \frac{cov(X,Y)}{var(X)}(X - E[X])$$

and

$$E[|Y - L[Y|X]|^2] = var(Y) - \frac{cov(X,Y)^2}{var(X)}.$$

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Here is a picture when $E[X] = 0, E[Y] = 0$:

Estimation Error: A Picture

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Here is a picture when $E[X] = 0, E[Y] = 0$:
Dimensions correspond to sample points, uniform sample space.

Estimation Error: A Picture

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Here is a picture when $E[X] = 0, E[Y] = 0$:
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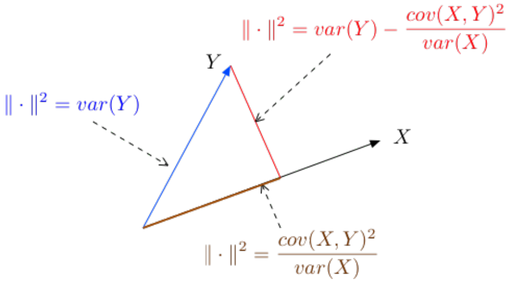

Vector Y at dimension ω is $\frac{1}{\sqrt{\Omega}} Y(\omega)$

Linear Regression Examples

Example 1:

Linear Regression Examples

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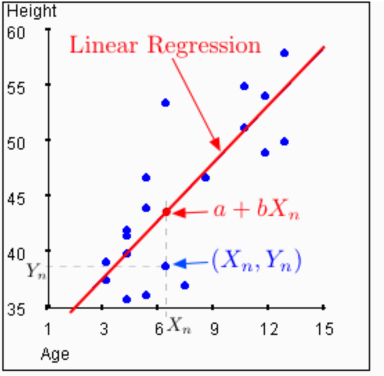

Linear Regression Examples

Example 2:

Linear Regression Examples

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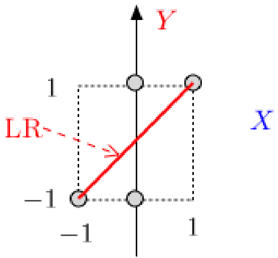

Linear Regression Examples

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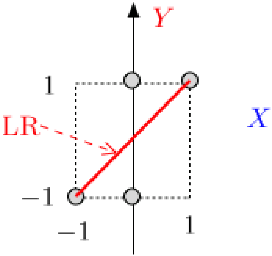
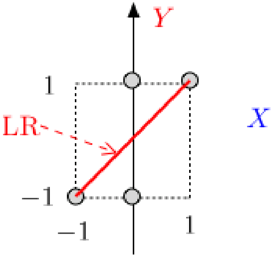


We find:

$E[X] =$

Linear Regression Examples

Example 2:



We find:

$$E[X] = 0;$$

Linear Regression Examples

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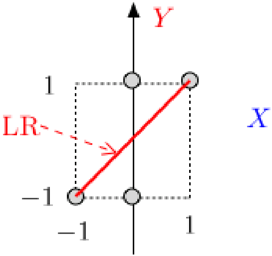

We find:

$$E[X] = 0; E[Y] =$$

Linear Regression Examples

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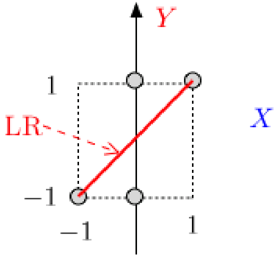
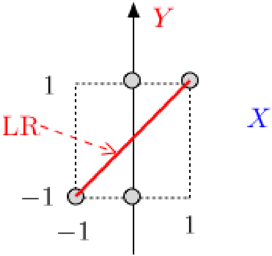


We find:

$$E[X] = 0; E[Y] = 0;$$

Linear Regression Examples

Example 2:



We find:

$$E[X] = 0; E[Y] = 0; E[X^2] =$$

Linear Regression Examples

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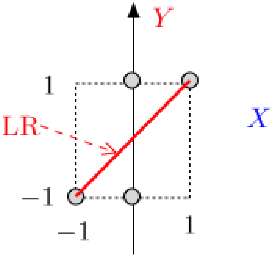

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$$E[X] = 0; E[Y] = 0; E[X^2] = 1/2;$$

Linear Regression Examples

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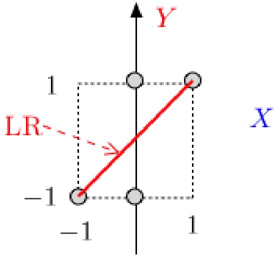

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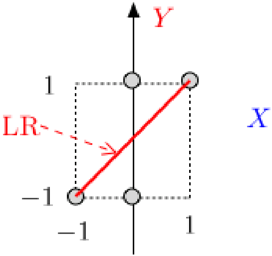

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Linear Regression Examples

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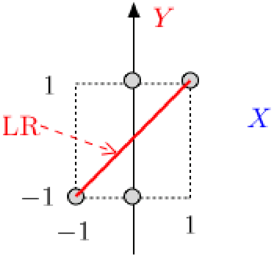

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$$E[X] = 0; E[Y] = 0; E[X^2] = 1/2; E[XY] = 1/2;$$
$$var[X] = E[X^2] - E[X]^2 =$$

Linear Regression Examples

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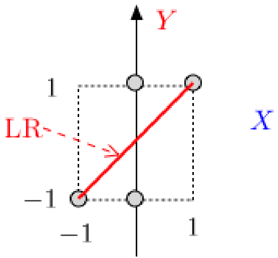

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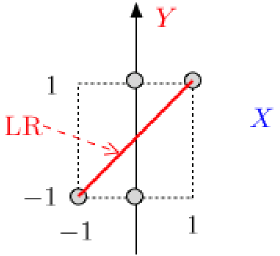

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Linear Regression Examples

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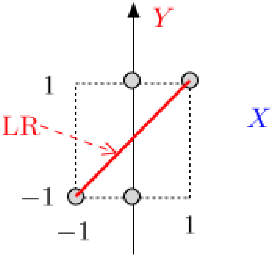

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Linear Regression Examples

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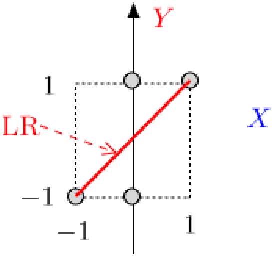

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LR: $\hat{Y} = E[Y] + \dfrac{cov(X, Y)}{var[X]}(X - E[X]) =$

Linear Regression Examples

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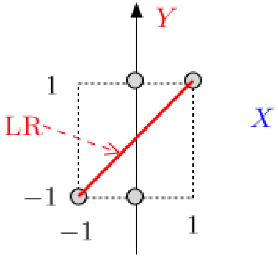

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Linear Regression Examples

Example 3:

Linear Regression Examples

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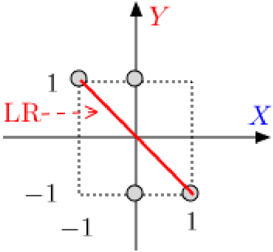

Linear Regression Examples

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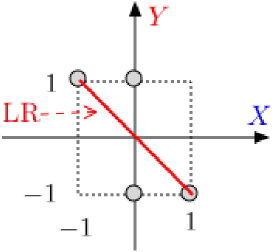
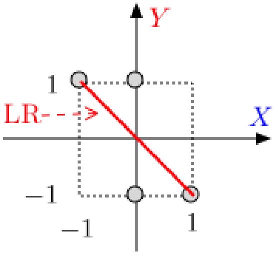


We find:

$E[X] =$

Linear Regression Examples

Example 3:



We find:

$E[X] = 0;$

Linear Regression Examples

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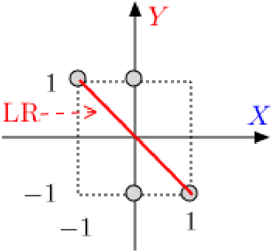

We find:

$$E[X] = 0; E[Y] =$$

Linear Regression Examples

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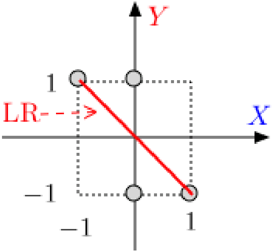

We find:

$E[X] = 0; E[Y] = 0;$

Linear Regression Examples

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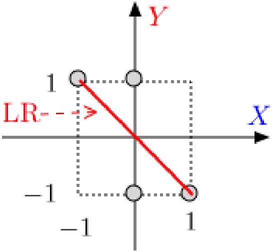

We find:

$$E[X] = 0; E[Y] = 0; E[X^2] =$$

Linear Regression Examples

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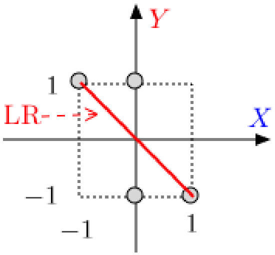

We find:

$$E[X] = 0; E[Y] = 0; E[X^2] = 1/2;$$

Linear Regression Examples

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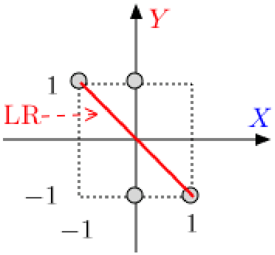

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Linear Regression Examples

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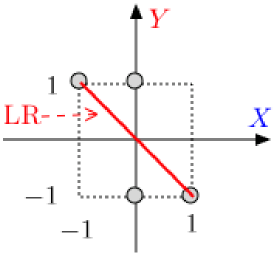

We find:

$$E[X] = 0; E[Y] = 0; E[X^2] = 1/2; E[XY] = -1/2;$$

Linear Regression Examples

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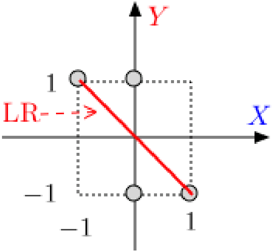

We find:

$$E[X] = 0; E[Y] = 0; E[X^2] = 1/2; E[XY] = -1/2;$$
$$var[X] = E[X^2] - E[X]^2 =$$

Linear Regression Examples

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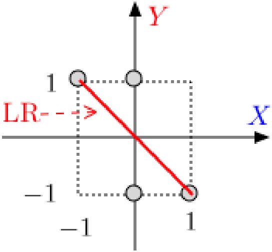

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Linear Regression Examples

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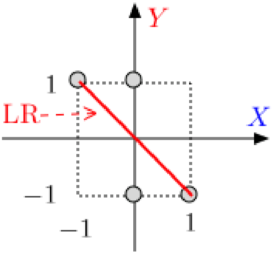

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$$E[X] = 0; E[Y] = 0; E[X^2] = 1/2; E[XY] = -1/2;$$
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Linear Regression Examples

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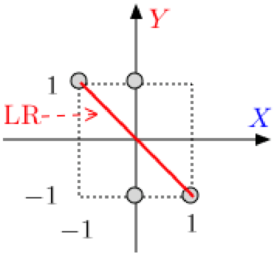

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Linear Regression Examples

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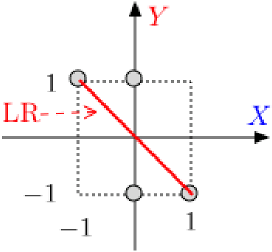

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$$\text{LR: } \hat{Y} = E[Y] + \frac{cov(X, Y)}{var[X]}(X - E[X]) =$$

Linear Regression Examples

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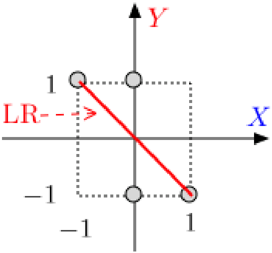

We find:

$$E[X] = 0; E[Y] = 0; E[X^2] = 1/2; E[XY] = -1/2;$$
$$var[X] = E[X^2] - E[X]^2 = 1/2; cov(X, Y) = E[XY] - E[X]E[Y] = -1/2;$$
$$\text{LR: } \hat{Y} = E[Y] + \frac{cov(X, Y)}{var[X]}(X - E[X]) = -X.$$

Linear Regression Examples

Example 4:

Linear Regression Examples

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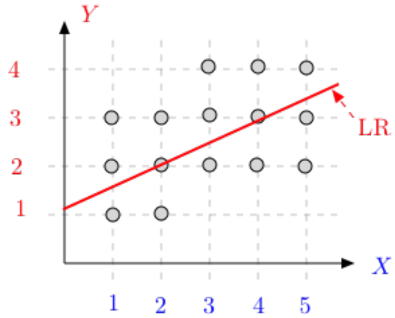
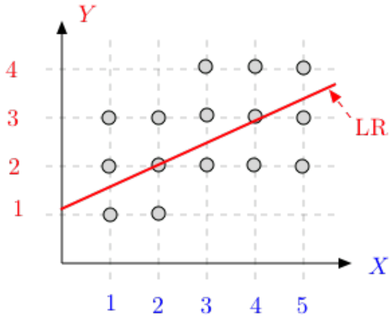


Linear Regression Examples

Example 4:

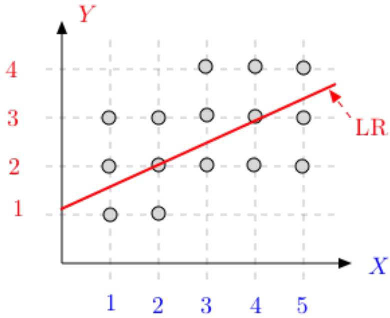


We find:

$$E[X] =$$

Linear Regression Examples

Example 4:

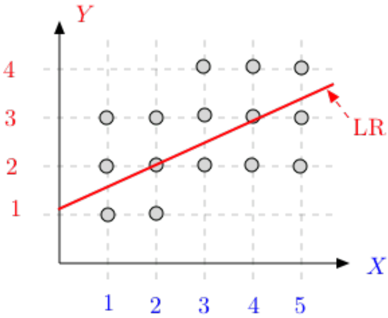


We find:

$$E[X] = 3;$$

Linear Regression Examples

Example 4:

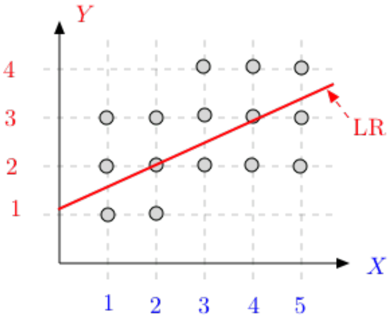


We find:

$E[X] = 3; E[Y] =$

Linear Regression Examples

Example 4:

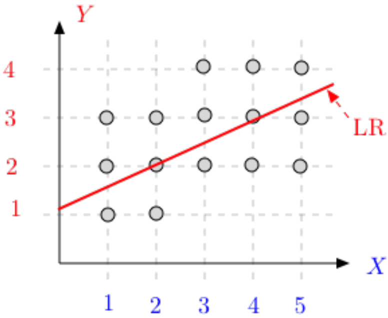


We find:

$$E[X] = 3; E[Y] = 2.5;$$

Linear Regression Examples

Example 4:

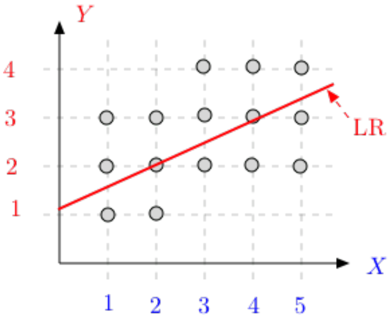


We find:

$$E[X] = 3; E[Y] = 2.5; E[X^2] = (3/15)(1 + 2^2 + 3^2 + 4^2 + 5^2) = 11;$$

Linear Regression Examples

Example 4:



We find:

$$E[X] = 3; E[Y] = 2.5; E[X^2] = (3/15)(1 + 2^2 + 3^2 + 4^2 + 5^2) = 11;$$
$$E[XY] = (1/15)(1 \times 1 + 1 \times 2 + \cdots + 5 \times 4) = 8.4;$$

Linear Regression Examples

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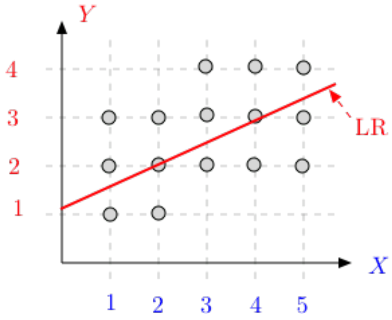

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$$var[X] = 11 - 9 = 2;$$

Linear Regression Examples

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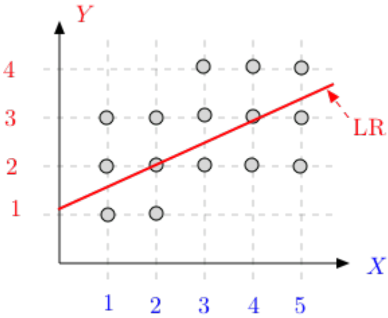

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Linear Regression Examples

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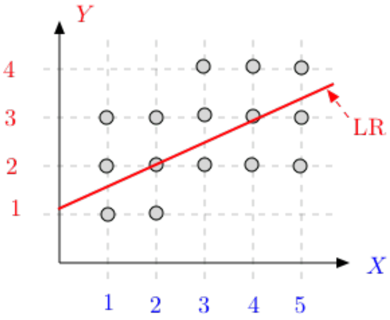

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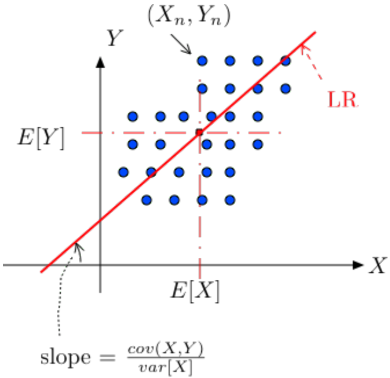
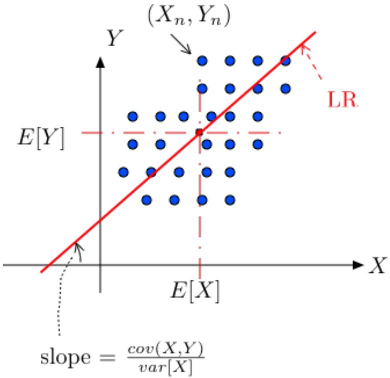
$$E[XY] = (1/15)(1 \times 1 + 1 \times 2 + \cdots + 5 \times 4) = 8.4;$$
$$var[X] = 11 - 9 = 2; cov(X, Y) = 8.4 - 3 \times 2.5 = 0.9;$$

LR: $\hat{Y} = 2.5 + \frac{0.9}{2}(X - 3) = 1.15 + 0.45X.$

LR: Another Figure



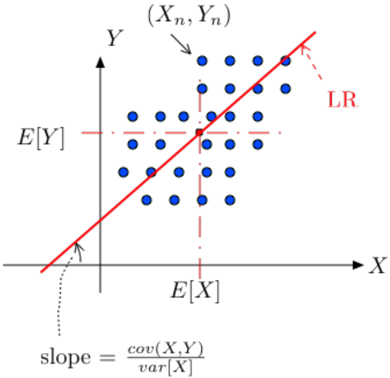
LR: Another Figure



Note that

- the LR line goes through $(E[X], E[Y])$

LR: Another Figure



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- its slope is $\frac{cov(X,Y)}{var(X)}$.

Quadratic Regression

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Statistics: Fix the assumption above.