Lecture 16: Continuing Probability.

Events, Conditional Probability, Independence, Bayes' Rule

Probability Basics:Poll

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(A) A set and a function on the elements.
(B) The values of the function are real numbers.
(C) The values of the function are positive integers.
(D) An element of the set is an outcome.
(E) There is an experiment associated with a probability space.
(F) The values in the set are integers.

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(A),(B), (D), (E).

Probability Basics Review

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    1. $0 \leq Pr[\omega] \leq 1$.
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  - **Events.**
    Event $A \subseteq \Omega$, $Pr[A] = \sum_{\omega \in \Omega} Pr[\omega]$.

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A, B, C, D

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(b) Either induction, or argue over sample points.

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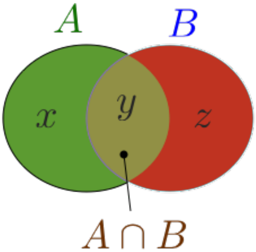

$$\begin{aligned} Pr[A] &= x + y \\ Pr[B] &= y + z \\ Pr[A \cap B] &= y \\ Pr[A \cup B] &= x + y + z \end{aligned}$$

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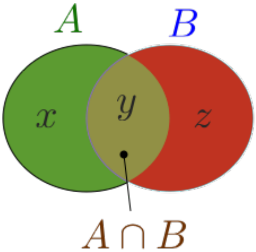

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Another view.

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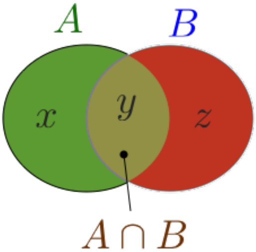

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Another view. Any $\omega \in A \cup B$ is in $A \cap \overline{B}$, $A \cup B$, or $\overline{A} \cap B$. So, add it up.

Roll a Red and a Blue Die.

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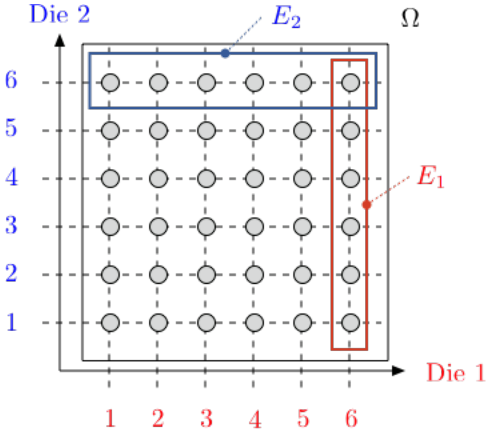

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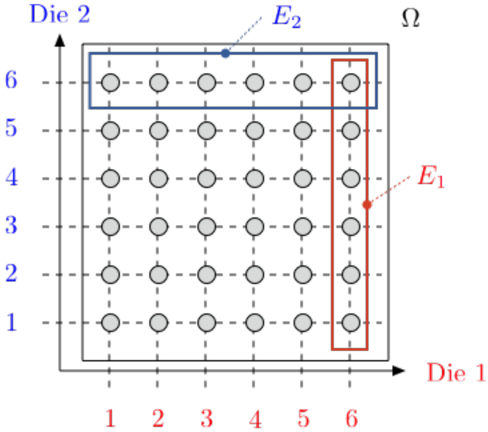

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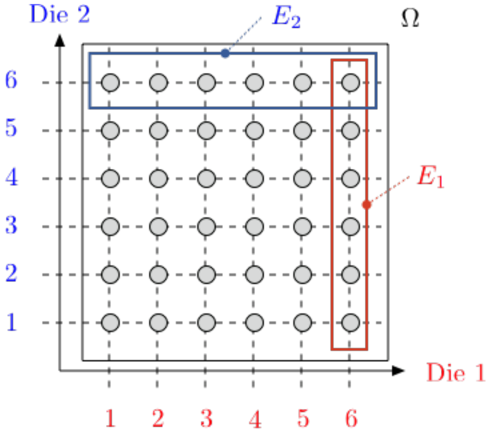

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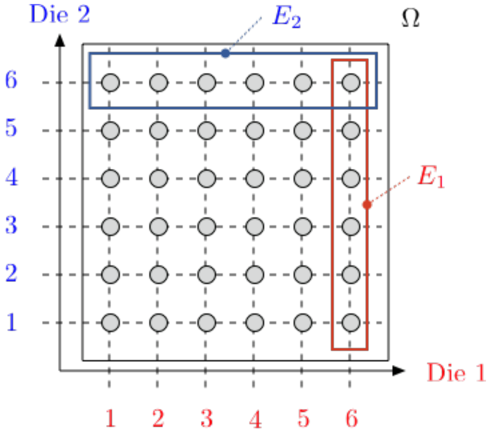

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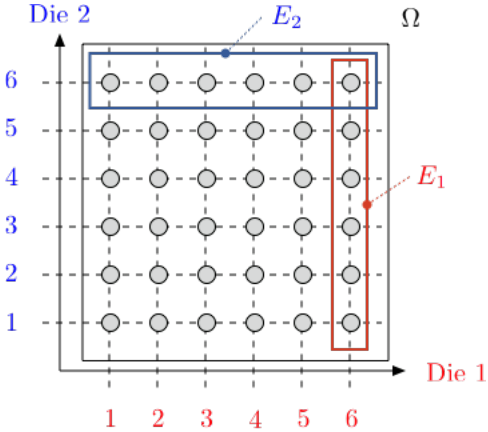

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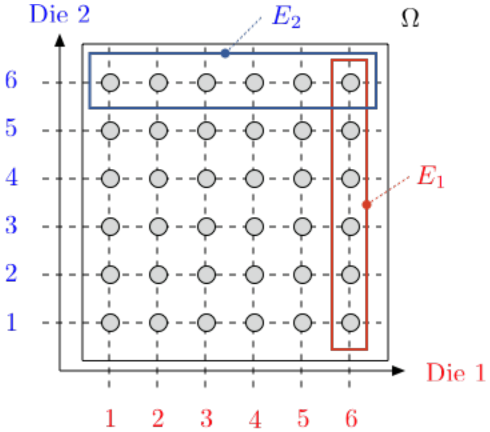

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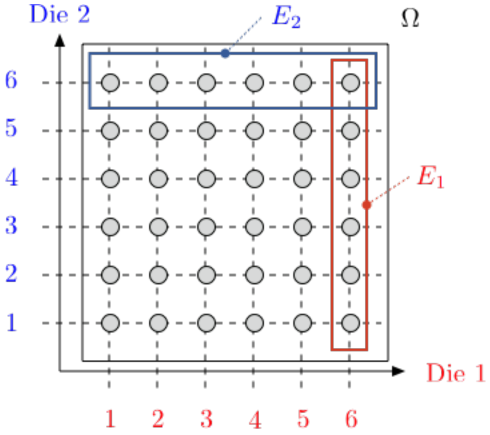

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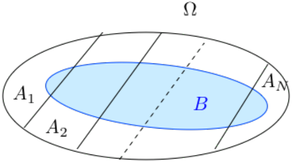
$$Pr[E_1] = \frac{6}{36}, Pr[E_2] = \frac{6}{36}, Pr[E_1 \cup E_2] = \frac{11}{36}.$$

Total probability

Assume that Ω is the union of the disjoint sets $A_1, \ldots, A_N$.



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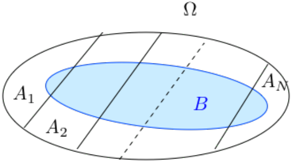

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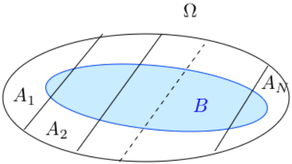

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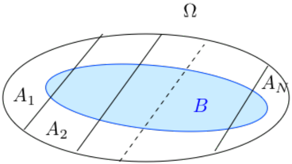

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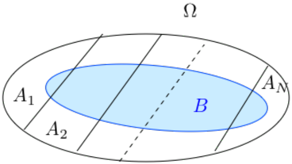

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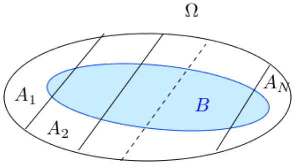

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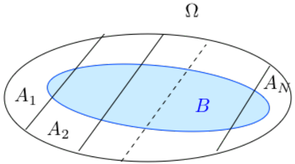

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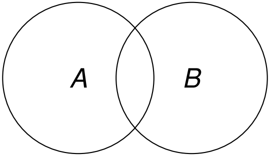
Add it up.

Conditional Probability.

Definition: The **conditional probability** of B given A is

$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}$$



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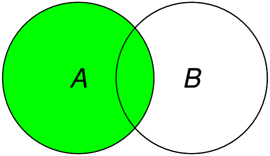

In A!

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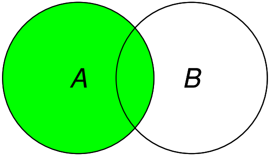

In A!
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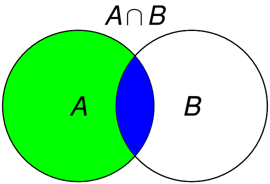

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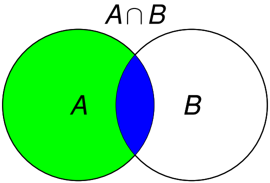

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Conditional probability: example.

Two coin flips.

Conditional probability: example.

Two coin flips. First flip is heads.

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Two coin flips. First flip is heads. Probability of two heads?

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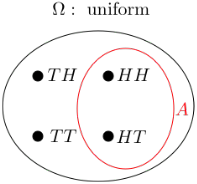

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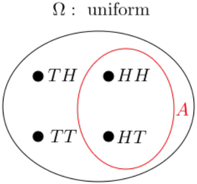

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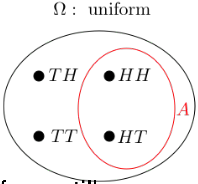

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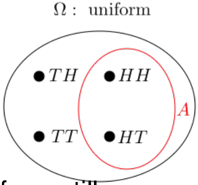

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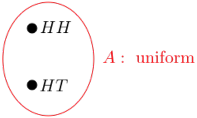

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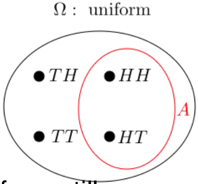

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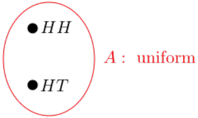

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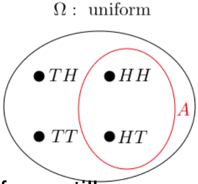

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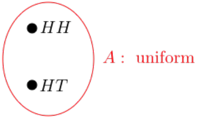

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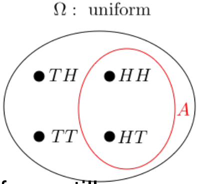

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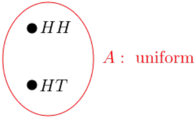

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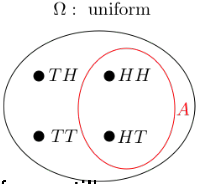

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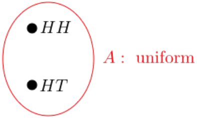

Event $B =$ two heads.

The probability of two heads if the first flip is heads.
The probability of B **given** A is $1/2$.

A similar example.

Two coin flips.

A similar example.

Two coin flips. At least one of the flips is heads.

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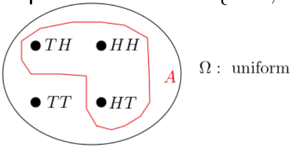

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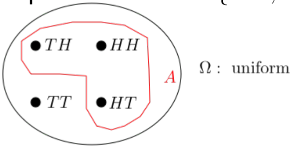

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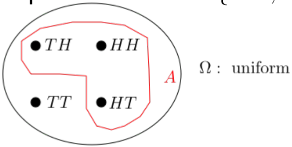

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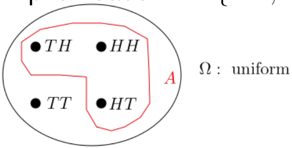

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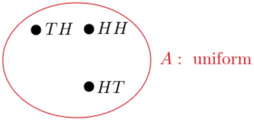

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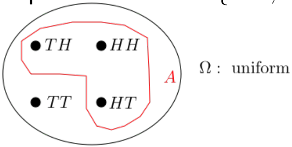

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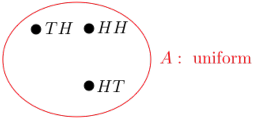

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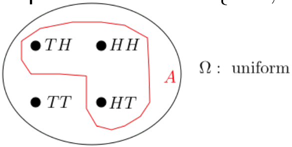

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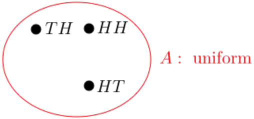

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The probability of two heads if at least one flip is heads.

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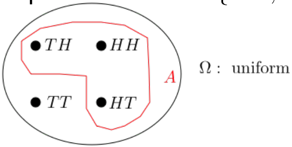

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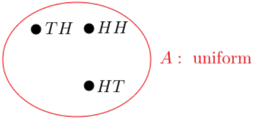

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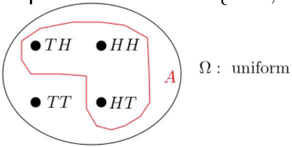

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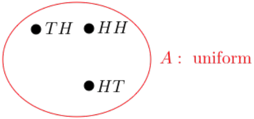

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Conditional Probability: A non-uniform example

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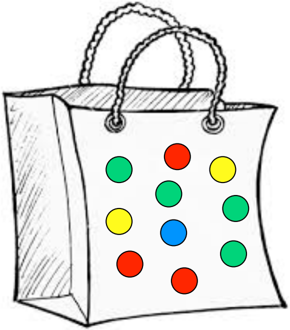
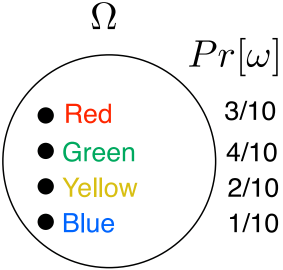
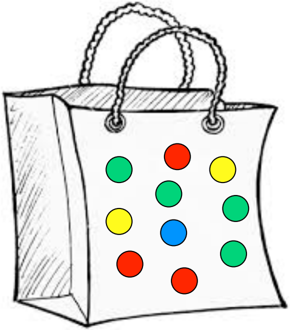


Physical experiment



Probability model

Conditional Probability: A non-uniform example



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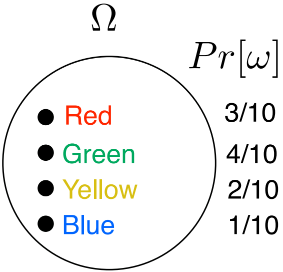

Probability model

$\Omega = \{$Red, Green, Yellow, Blue$\}$

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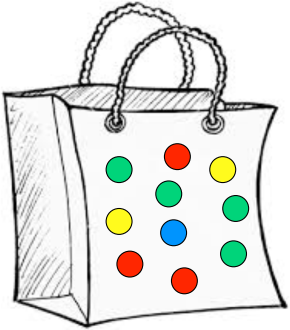

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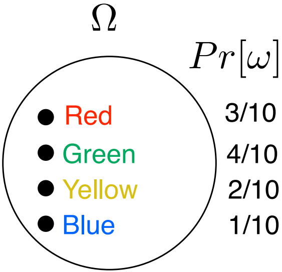

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$\Omega = \{$Red, Green, Yellow, Blue$\}$

$Pr[\text{Red}|\text{Red or Green}] =$

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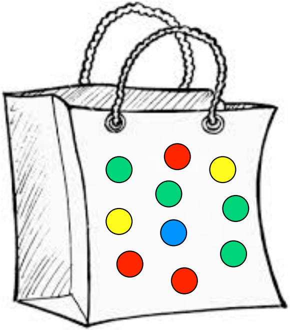

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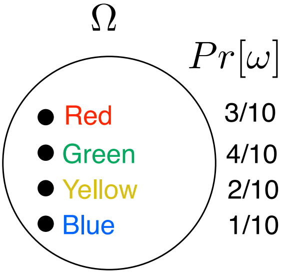
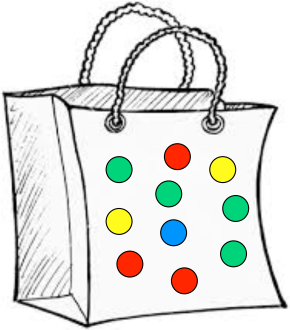


Probability model

$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

$$Pr[\text{Red}|\text{Red or Green}] = \frac{3}{7} =$$

Conditional Probability: A non-uniform example



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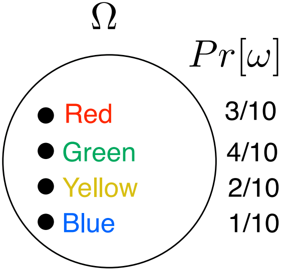

Probability model

$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

$$Pr[\text{Red}|\text{Red or Green}] = \frac{3}{7} = \frac{Pr[\text{Red} \cap (\text{Red or Green})]}{Pr[\text{Red or Green}]}$$

Another non-uniform example

Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$.

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Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$.
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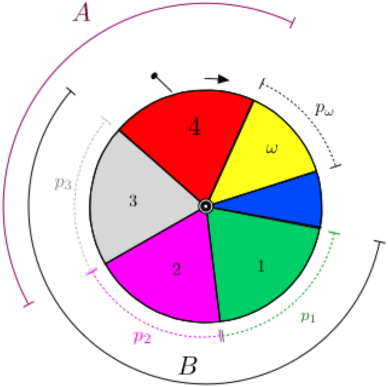

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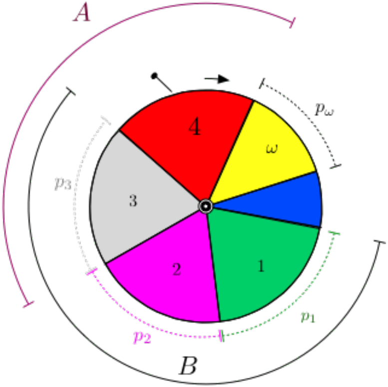

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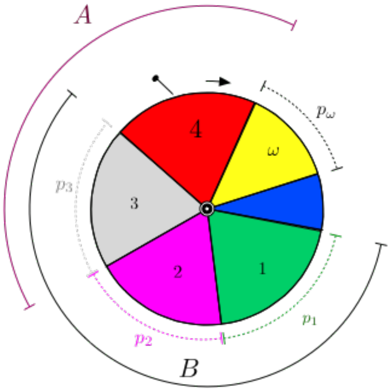

$$Pr[A|B] = \frac{p_3}{p_1 + p_2 + p_3} = \frac{Pr[A \cap B]}{Pr[B]}.$$

Yet another non-uniform example

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Yet another non-uniform example

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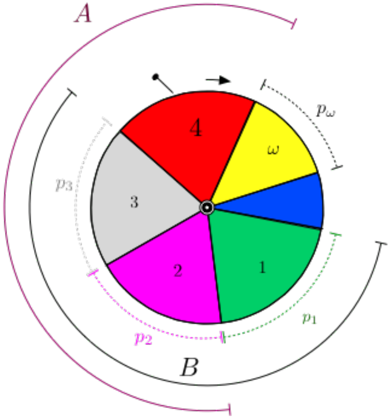

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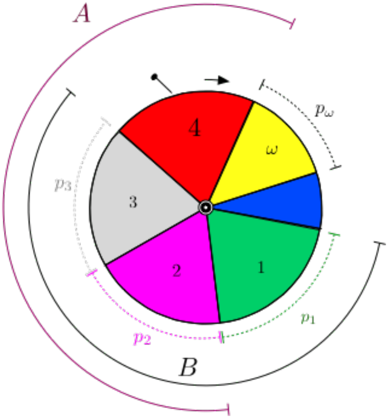

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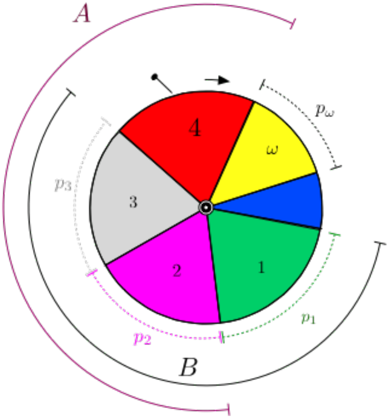

$$Pr[A|B] = \frac{p_2 + p_3}{p_1 + p_2 + p_3} = \frac{Pr[A \cap B]}{Pr[B]}.$$

More fun with conditional probability.

Toss a red and a blue die, sum is 4,

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Toss a red and a blue die, sum is 4,
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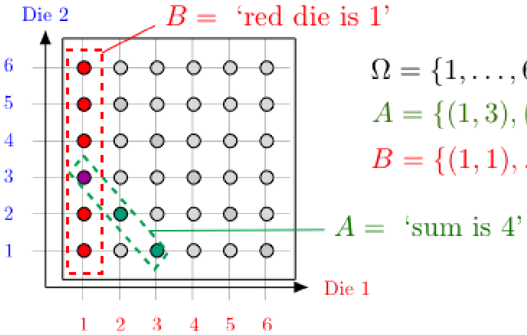

$\Omega = \{1, \ldots, 6\}^2$

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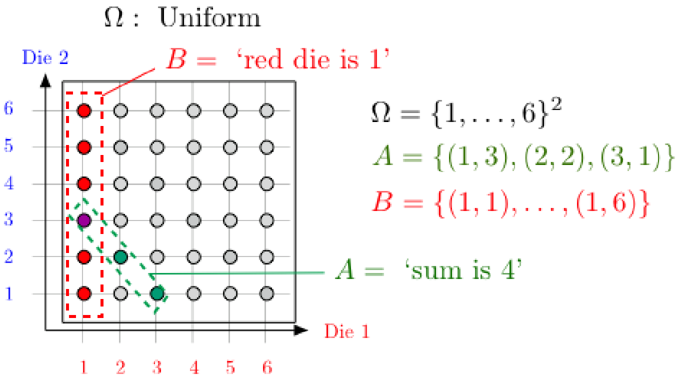

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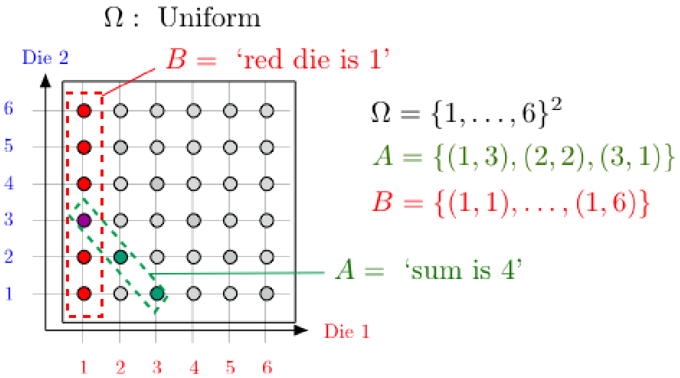

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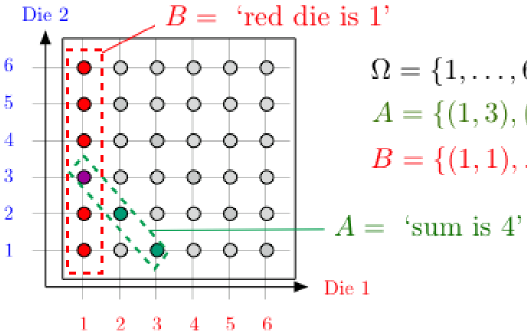

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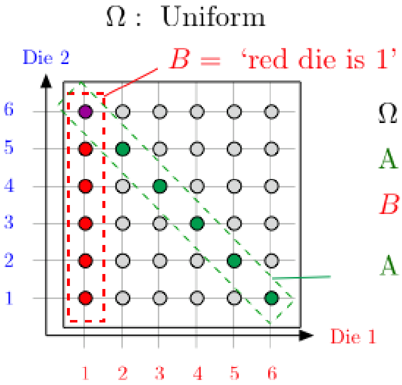

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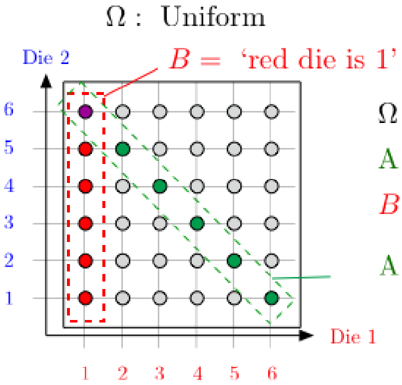

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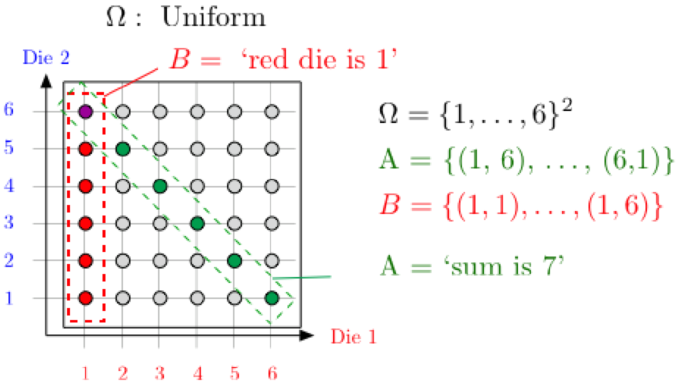

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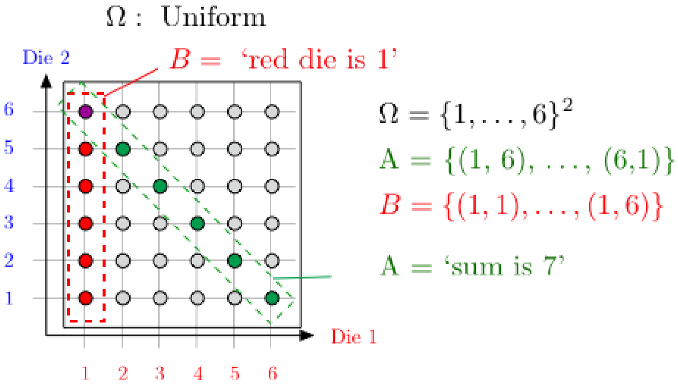

$Pr[B|A] = \frac{|B \cap A|}{|A|} = \frac{1}{6}$; versus $Pr[B] = \frac{1}{6}$.

Observing *A* does not change your mind about the likelihood of *B*.

Such empty: poll

Suppose I toss 3 balls into 3 bins.

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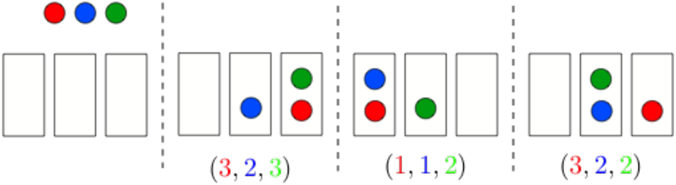

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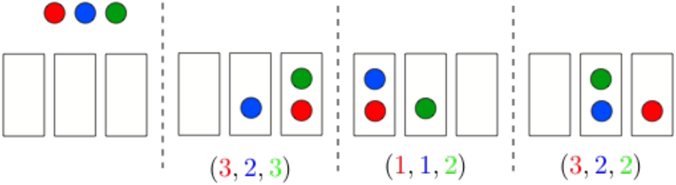

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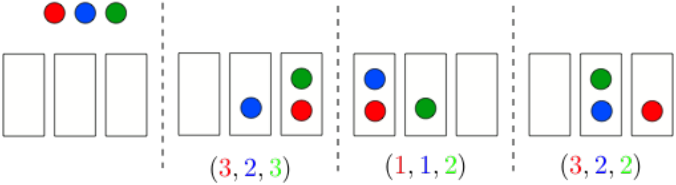

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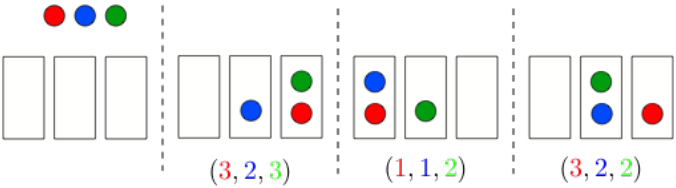

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(A) 1/27
(B) 8/27
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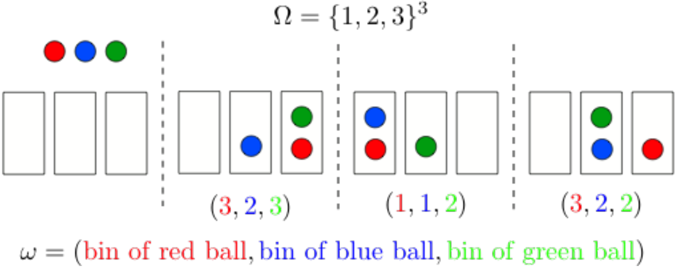

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Next slide.

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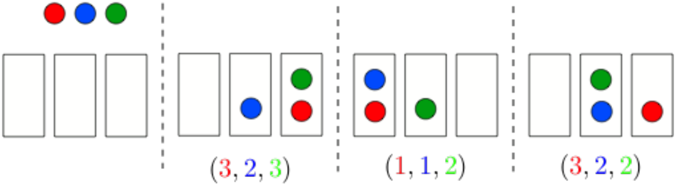

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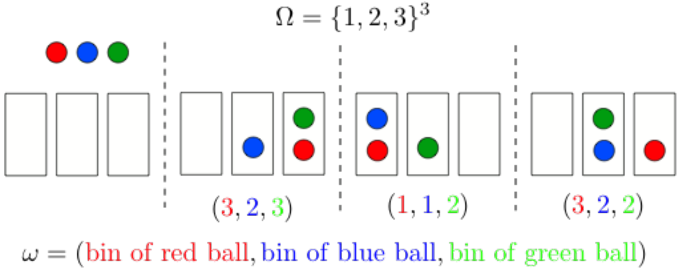

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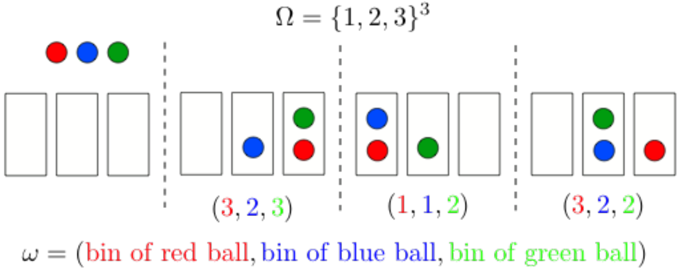

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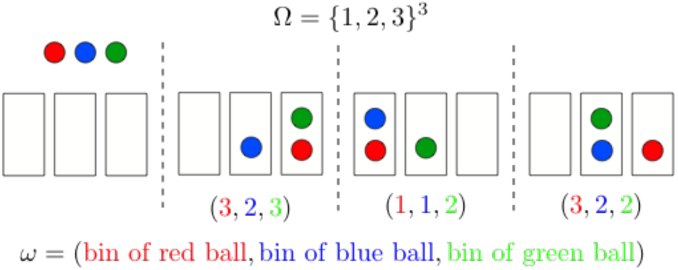

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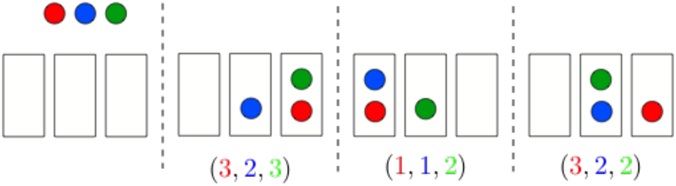

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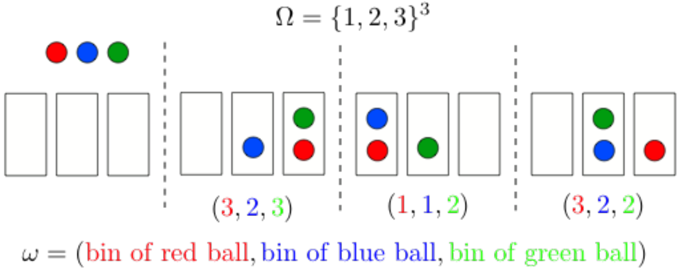

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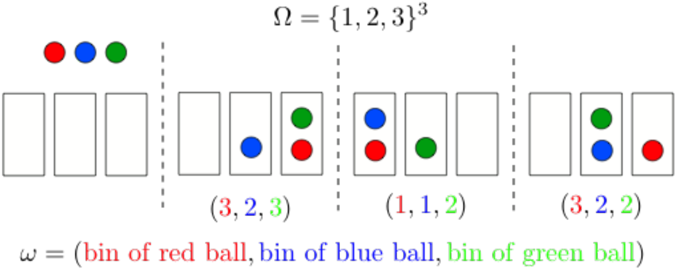

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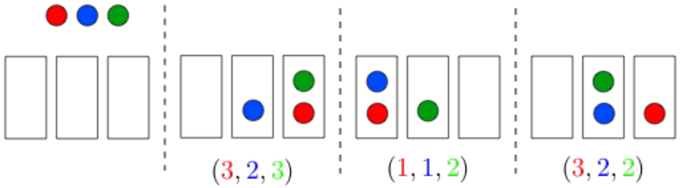

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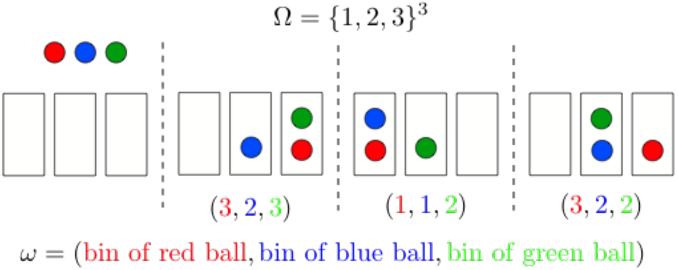

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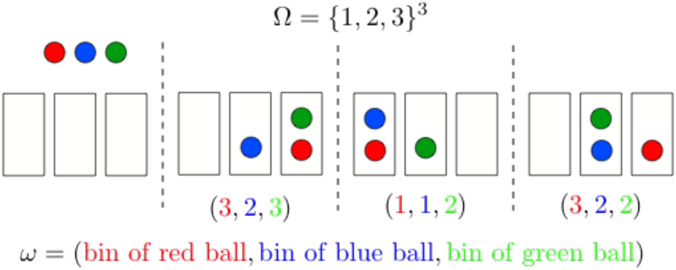

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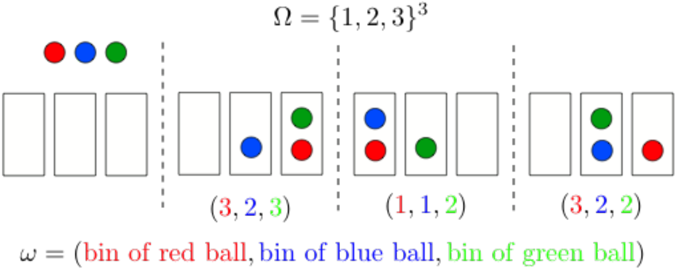

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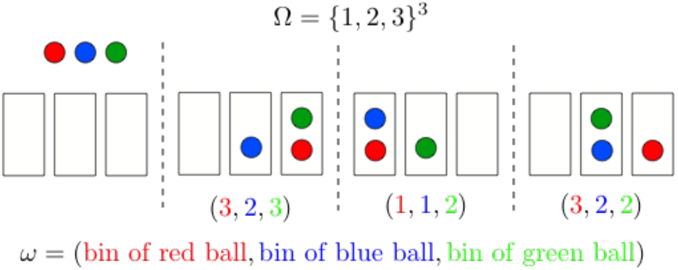

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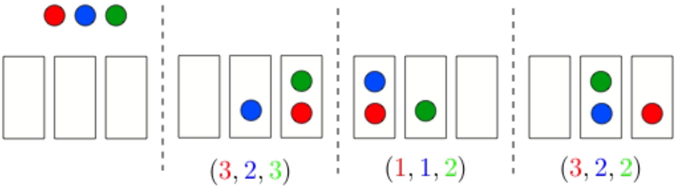

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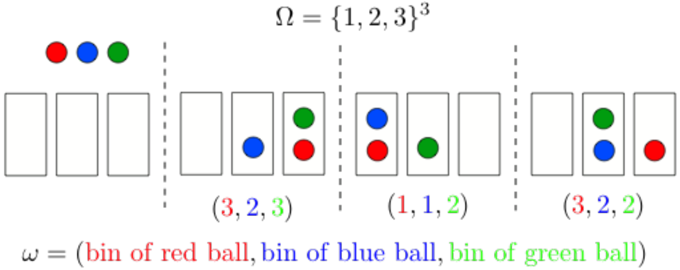

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A is less likely given B: If second bin is empty the first is more likely to have balls in it.

Gambler's fallacy.

Flip a fair coin 51 times.

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The likelihood of 51st heads does not depend on the previous flips.

Product Rule

Recall the definition:

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Theorem Product Rule
Let $A_1, A_2, \ldots, A_n$ be events. Then

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so that the result holds for $n+1$. □

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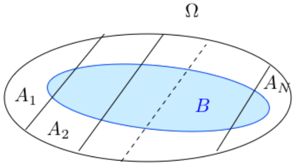
More about such questions later. For fun, check "N. Taleb: Fooled by randomness."

Total probability with Conditional Probability.

Assume that Ω is the union of the disjoint sets $A_1, \ldots, A_N$.



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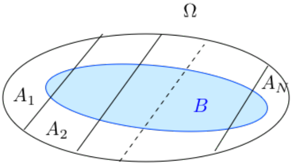

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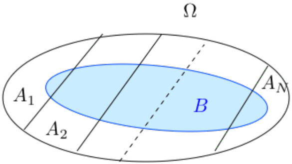

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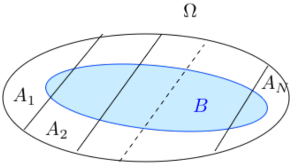

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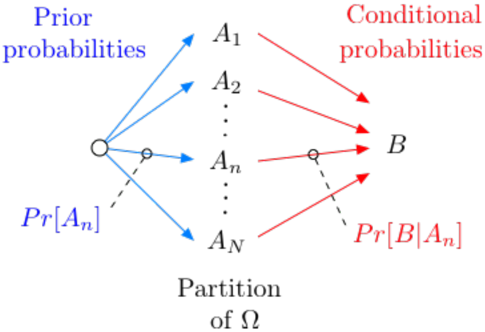

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Conditional Probability: Pictures

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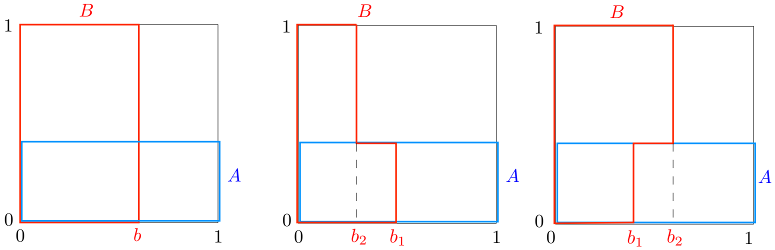
Illustrations: Pick a point uniformly in the unit square



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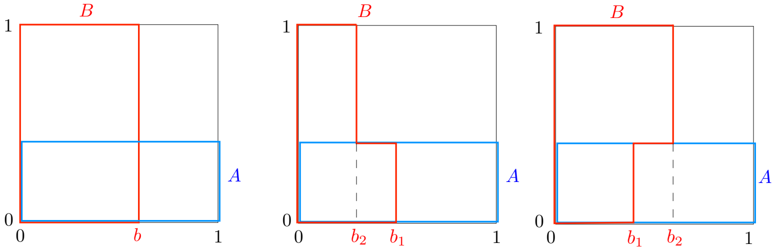

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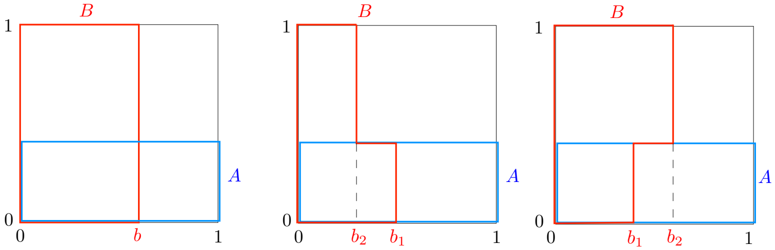

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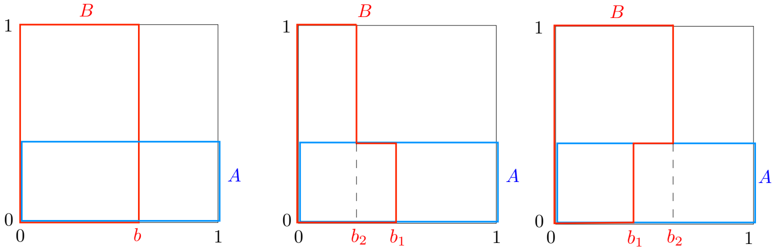

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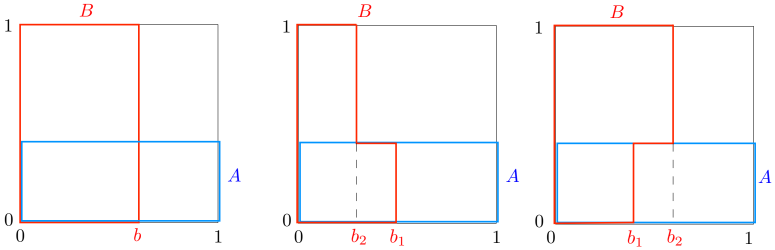

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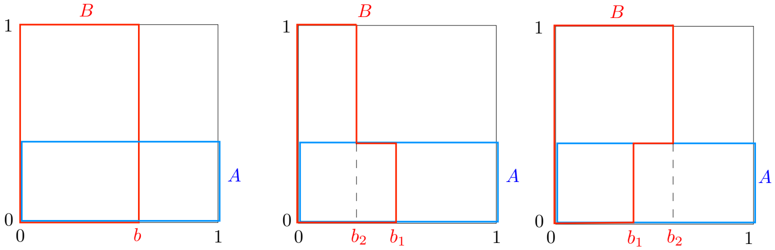

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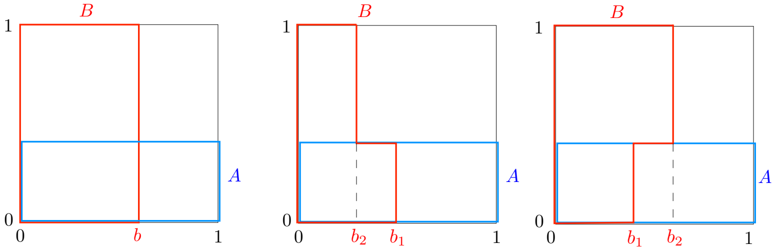

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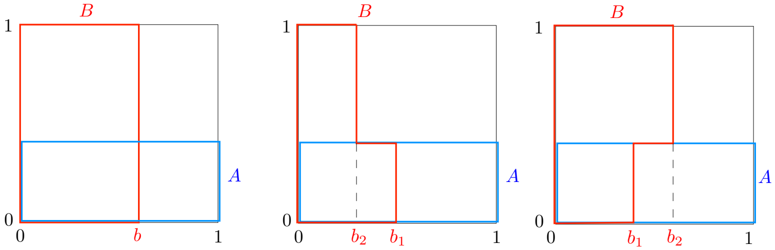

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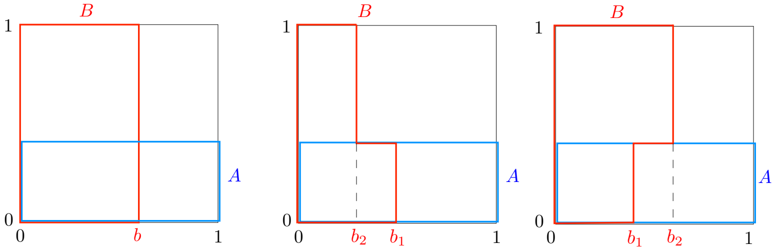

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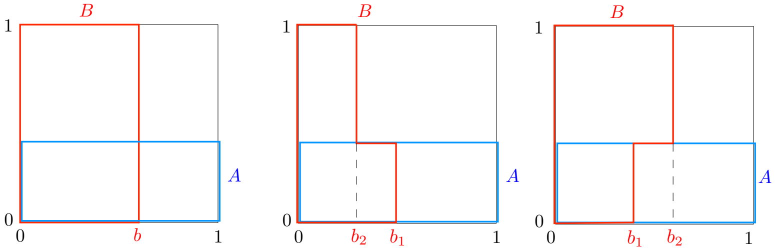

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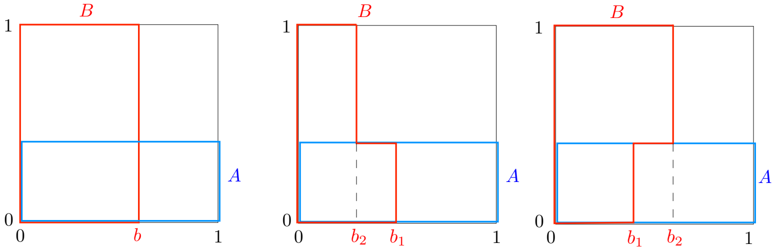

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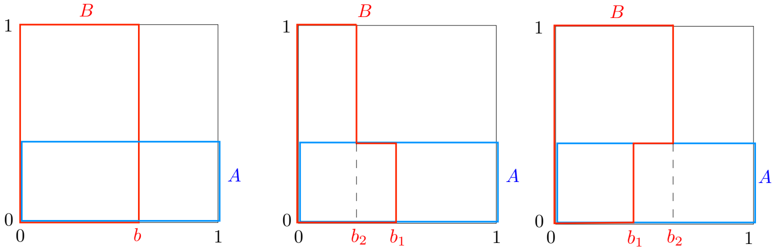

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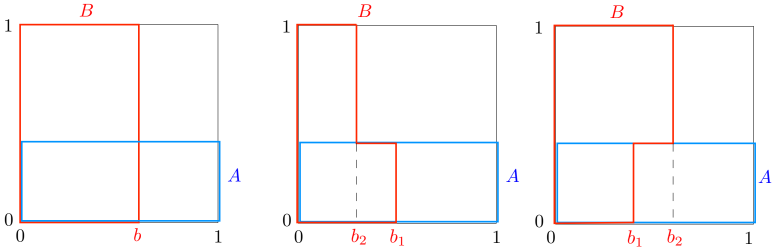

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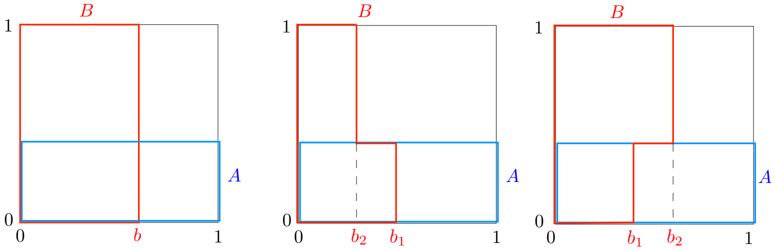

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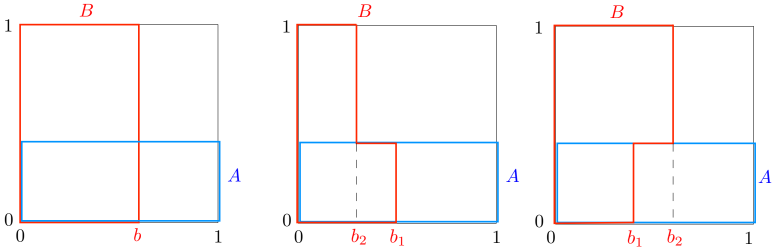

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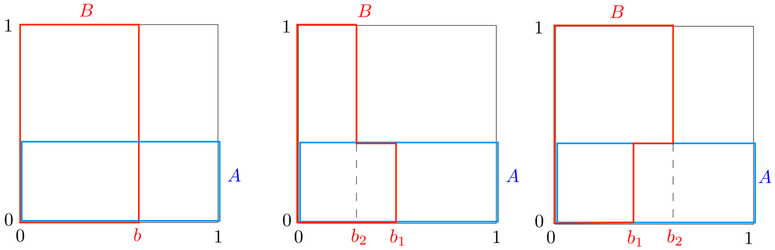

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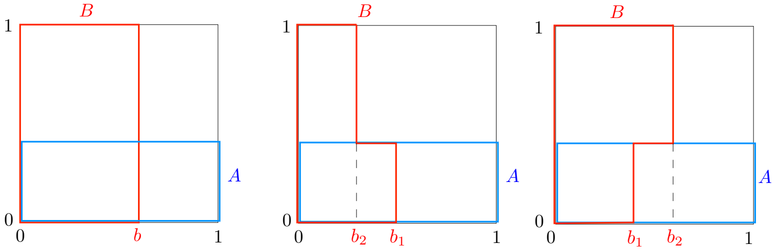

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