Stars and Bars Poll

Mark whats correct.

(A) ways to split n dollars among k: $\binom{n+k-1}{k-1}$

(B) ways to split k dollars among n: $\binom{k+n-1}{n-1}$

(C) ways to split 5 dollars among 3: $\binom{7}{5}$

(D) ways to split 5 dollars among 3: $\binom{5+3-1}{3-1}$

All correct.

Poll

Mark whats corect.

(A) $|10 \text{ digit numbers}| = 10^{10}$

(B) $|k \text{ coin tosses}| = 2^k$

(C) $|10 \text{ digit numbers}| = 9 * 10^9$

(D) $|n \text{ digit base } m \text{ numbers}| = m^n$

(E) $|n \text{ digit base } m \text{ numbers}| = (m-1)m^{n-1}$

(A) or (C)? (D) or (E)? (B) are correct.

Refresh: Counting.

First Rule of counting: Objects from a sequence of choices:

n_i possibilities for ith choice.

$n_1 \times n_2 \times \cdots \times n_k$ objects.

Second Rule of counting: If order does not matter.

Count with order. Divide by number of orderings/sorted object.

Typically: $\binom{n}{k}$.

Stars and Bars: Sample k objects with replacement from n.

Order doesn't matter. k stars $n-1$ bars.

Typically: $\binom{n+k-1}{k}$ or $\binom{n+k-1}{n-1}$.

Inclusion/Exclusion: two sets of objects.

Add number of each and then subtract intersection of sets.

Sum Rule: If disjoint just add.

Combinatorial Proofs: Identity from counting same in two ways.

Pascal's Triangle Example: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

RHS: Number of subsets of $n+1$ items size k.

LHS: $\binom{n}{k-1}$ counts subsets of $n+1$ items with first item.

$\binom{n}{k}$ counts subsets of $n+1$ items without first item.

Disjoint – so add!

CS70: On to probability.

Modeling Uncertainty: Probability Space

1. Key Points
2. Random Experiments
3. Probability Space

Key Points

- Uncertainty does not mean "nothing is known"

- How to best make decisions under uncertainty?

  - Buy stocks
  - Detect signals (transmitted bits, speech, images, radar, diseases, etc.)
  - Control systems (Internet, airplane, robots, self-driving cars, schedule surgeries in a hospital, etc.)

- How to best use 'artificial' uncertainty?

  - Play games of chance
  - Design randomized algorithms.

- Probability

  - Models knowledge about uncertainty
  - Optimizes use of knowledge to make decisions

The Magic of Probability

Uncertainty: vague, fuzzy, confusing, scary, hard to think about.

Probability:
Precise, unambiguous, simple(!) way to reason about uncertainty.

Uncertainty = Fear

Probability = Serenity

Our mission: help you discover the serenity of Probability, i.e., enable you to think clearly about uncertainty.

Your cost: focused attention and practice on examples and problems.

Random Experiment: Flip one Fair Coin

Flip a fair coin: (*One flips or tosses a coin*)

- ▶ Possible outcomes: Heads (H) and Tails (T)
  (*One flip yields either 'heads' or 'tails'.*)
- ▶ Likelihoods: $H : 50\%$ and $T : 50\%$

Random Experiment: Flip one Fair Coin

Flip a fair coin:

What do we mean by the likelihood of tails is 50%?

Two interpretations:

▶ Single coin flip: 50% chance of 'tails' [subjectivist]

Willingness to bet on the outcome of a single flip

▶ Many coin flips: About half yield 'tails' [frequentist]

Makes sense for many flips

▶ Question: Why does the fraction of tails converge to the same value every time? Statistical Regularity! Deep!

Random Experiment: Flip one Fair Coin

Flip a fair coin: model
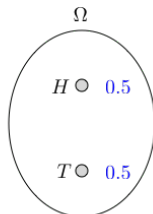
Physical Experiment   Probability Model

▶ The physical experiment is complex. (Shape, density, initial momentum and position, ...)

▶ The Probability model is simple:

    ▶ A set Ω of outcomes: $\Omega = \{H, T\}$.

    ▶ A probability assigned to each outcome:
    $Pr[H] = 0.5, Pr[T] = 0.5$.

Random Experiment: Flip one Unfair Coin
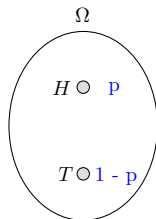
Flip an unfair (biased, loaded) coin:

H: 45%

T: 55%

- ▶ Possible outcomes: Heads (H) and Tails (T)
- ▶ Likelihoods: $H : p \in (0, 1)$ and $T : 1 - p$
- ▶ Frequentist Interpretation:

  Flip many times $\Rightarrow$ Fraction $1 - p$ of tails

- ▶ Question: How can one figure out p? Flip many times
- ▶ Tautology? No: Statistical regularity!

Random Experiment: Flip one Unfair Coin

Flip an unfair (biased, loaded) coin: model

Physical Experiment

Probability Model

Flip Two Fair Coins

- Possible outcomes: $\{HH, HT, TH, TT\} \equiv \{H, T\}^2$.
- Note: $A \times B := \{(a, b) \mid a \in A, b \in B\}$ and $A^2 := A \times A$.
- Likelihoods: $1/4$ each.

Flip Glued Coins

Flips two coins glued together side by side:

Glued coins

50%

50%

- ▶ Possible outcomes: $\{HT, TH\}$.
- ▶ Likelihoods: $HT : 0.5, TH : 0.5$.
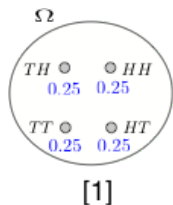- ▶ Note: Coins are glued so that they show different faces.

Flip two Attached Coins

Flips two coins attached by a spring:

Attached coins

- ▶ Possible outcomes: $\{HH, HT, TH, TT\}$.
- ▶ Likelihoods: $HH : 0.4, HT : 0.1, TH : 0.1, TT : 0.4$.
- ▶ Note: Coins are attached so that they tend to show the same face, unless the spring twists enough.
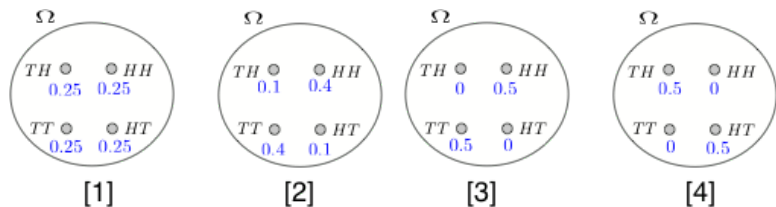
Flipping Two Coins

Here is a way to summarize the four random experiments:
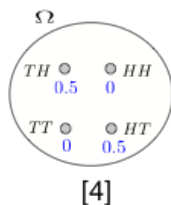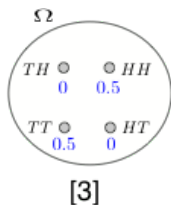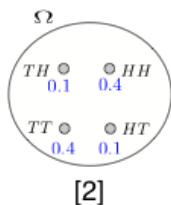
- ▶ Ω is the set of *possible* outcomes;
- ▶ Each outcome has a probability (likelihood);
- ▶ The probabilities are ≥ 0 and add up to 1;
- ▶ Fair coins: [1]; Glued coins: [3], [4];

    Spring-attached coins: [2];

Flipping Two Coins

Important remarks:

► Each outcome describes the two coins.

► E.g., *HT* is one outcome of each of the above experiments.

► Wrong to think that outcomes are $\{H, T\}$ and that one picks twice from that set.

► Indeed, this viewpoint misses the relationship between the two flips.

► Each $\omega \in \Omega$ describes one outcome of the complete experiment.

► Ω and the probabilities specify the random experiment.

Flipping *n* times
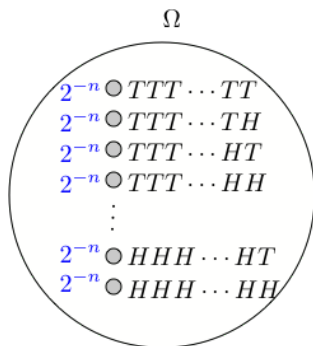
Flip a fair coin *n* times (some $n \geq 1$):

- Possible outcomes: $\{TT \cdots T, TT \cdots H, \ldots, HH \cdots H\}$.

  Thus, 2^n possible outcomes.

- Note: $\{TT \cdots T, TT \cdots H, \ldots, HH \cdots H\} = \{H, T\}^n$.

  $A^n := \{(a_1, \ldots, a_n) \mid a_1 \in A, \ldots, a_n \in A\}$. $|A^n| = |A|^n$.
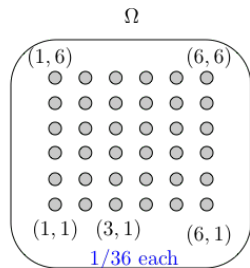
- Likelihoods: $1/2^n$ each.

$$\Omega$$

$2^{-n} \bigcirc TTT \cdots TT$
$2^{-n} \bigcirc TTT \cdots TH$
$2^{-n} \bigcirc TTT \cdots HT$
$2^{-n} \bigcirc TTT \cdots HH$
$\vdots$
$2^{-n} \bigcirc HHH \cdots HT$
$2^{-n} \bigcirc HHH \cdots HH$

Roll two Dice

Roll a balanced 6-sided die twice:

- ▶ Possible outcomes: $\{1,2,3,4,5,6\}^2 = \{(a,b) \mid 1 \le a, b \le 6\}$.
- ▶ Likelihoods: $1/36$ for each.

Physical Experiment   Probability Model

Probability Space.

1. A "random experiment":
   - (a) Flip a biased coin;
   - (b) Flip two fair coins;
   - (c) Deal a poker hand.

2. A set of possible outcomes: Ω.
   - (a) $\Omega = \{H, T\}$;
   - (b) $\Omega = \{HH, HT, TH, TT\}$; $|\Omega| = 4$;
   - (c) $\Omega = \{\ A\spadesuit\ A\diamondsuit\ A\clubsuit\ A\heartsuit\ K\spadesuit,\ \ A\spadesuit\ A\diamondsuit\ A\clubsuit\ A\heartsuit\ Q\spadesuit, \ldots\}$
     $|\Omega| = \binom{52}{5}$.

3. Assign a probability to each outcome: $Pr : \Omega \to [0, 1]$.
   - (a) $Pr[H] = p, Pr[T] = 1 - p$ for some $p \in [0, 1]$
   - (b) $Pr[HH] = Pr[HT] = Pr[TH] = Pr[TT] = \frac{1}{4}$
   - (c) $Pr[\ A\spadesuit\ A\diamondsuit\ A\clubsuit\ A\heartsuit\ K\spadesuit\ ] = \cdots = 1/\binom{52}{5}$

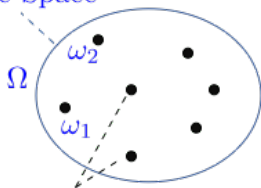# Probability Space: formalism.

Ω is the **sample space.**
$\omega \in \Omega$ is a **sample point**. (Also called an **outcome**.)
Sample point ω has a probability $Pr[\omega]$ where

- $0 \le Pr[\omega] \le 1$;

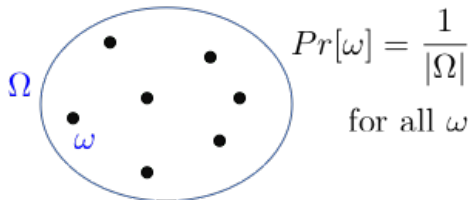- $\sum_{\omega \in \Omega} Pr[\omega] = 1$.

$$0 \le Pr[\omega] \le 1$$

$$\sum_{\omega} Pr[\omega] = 1$$

Probability Space: Formalism.

In a **uniform probability space** each outcome ω is equally probable: $Pr[\omega] = \frac{1}{|\Omega|}$ for all $\omega \in \Omega$.

Uniform Probability Space

$$Pr[\omega] = \frac{1}{|\Omega|}$$
for all ω

Examples:

▶ Flipping two fair coins, dealing a poker hand are uniform probability spaces.

▶ Flipping a biased coin is not a uniform probability space.

Probability Space: Formalism

Simplest physical model of a uniform probability space:
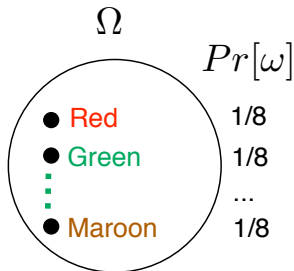
Physical experiment   Probability model

A bag of identical balls, except for their color (or a label). If the bag is well shaken, every ball is equally likely to be picked.

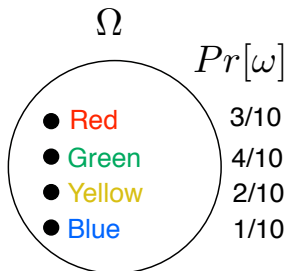$$\Omega = \{\text{white, red, yellow, grey, purple, blue, maroon, green}\}$$
$$Pr[\text{blue}] = \frac{1}{8}.$$

Probability Space: Formalism

Simplest physical model of a non-uniform probability space:

Physical experiment

$$\Omega$$
$$Pr[\omega]$$
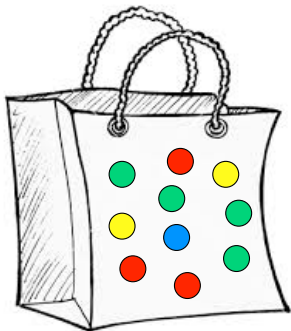
- Red   3/10
- Green   4/10
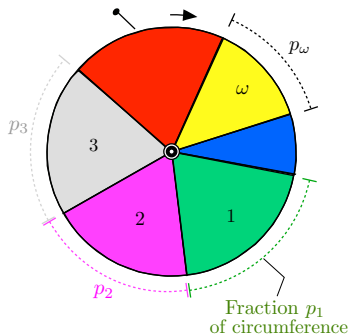- Yellow   2/10
- Blue   1/10

Probability model

$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$
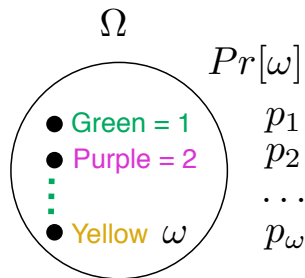$$Pr[\text{Red}] = \frac{3}{10}, Pr[\text{Green}] = \frac{4}{10}, \text{ etc.}$$

Note: Probabilities are restricted to rational numbers: $\frac{N_k}{N}$.

Probability Space: Formalism

Physical model of a general non-uniform probability space:

Physical experiment   Probability model

The roulette wheel stops in sector ω with probability p_ω.

$$\Omega = \{1, 2, 3, \ldots, N\}, Pr[\omega] = p_\omega.$$

An important remark

- ▶ The random experiment selects one and only one outcome in Ω.

- ▶ For instance, when we flip a fair coin twice

    - ▶ $\Omega = \{HH, TH, HT, TT\}$
    - ▶ The experiment selects *one* of the elements of Ω.

- ▶ In this case, its wrong to think that $\Omega = \{H, T\}$ and that the experiment selects two outcomes.

- ▶ Why? Because this would not describe how the two coin flips are related to each other.

- ▶ For instance, say we glue the coins side-by-side so that they face up the same way. Then one gets *HH* or *TT* with probability 50% each. This is not captured by 'picking two outcomes.'

Summary of Probability Basics

Modeling Uncertainty: Probability Space

1. Random Experiment
2. Probability Space: $\Omega; Pr[\omega] \in [0,1]; \sum_\omega Pr[\omega] = 1$.
3. Uniform Probability Space: $Pr[\omega] = 1/|\Omega|$ for all $\omega \in \Omega$.

Onwards in Probability.

Events, Conditional Probability, Independence, Bayes' Rule

CS70: On to Events.

Events, Conditional Probability, Independence, Bayes' Rule

Today: Events.

Probability Basics Review

Setup:

- ▶ Random Experiment.
  Flip a fair coin twice.

- ▶ Probability Space.

  - ▶ **Sample Space:** Set of outcomes, Ω.
    $\Omega = \{HH, HT, TH, TT\}$
    (Note: Not $\Omega = \{H, T\}$ with two picks!)

  - ▶ **Probability:** $Pr[\omega]$ for all $\omega \in \Omega$.
    $Pr[HH] = \cdots = Pr[TT] = 1/4$
    1. $0 \le Pr[\omega] \le 1$.
    2. $\sum_{\omega \in \Omega} Pr[\omega] = 1$.

Set notation review
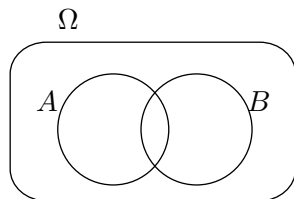
Figure: Two events
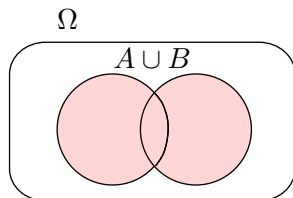
Figure: Union (or)
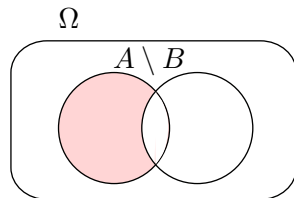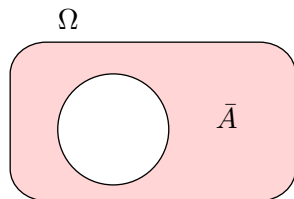
Figure: Difference (A, not B)
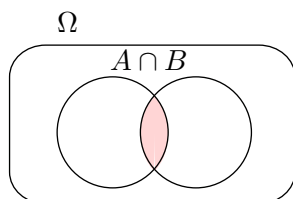
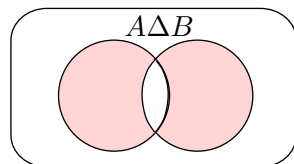Figure: Complement (not)

Figure: Intersection (and)
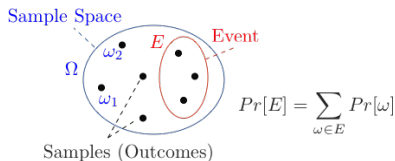
Figure: Symmetric difference (only one)

Probability of exactly one 'heads' in two coin flips?

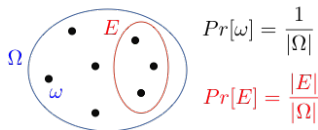Idea: Sum the probabilities of all the different outcomes that have exactly one 'heads': HT, TH.
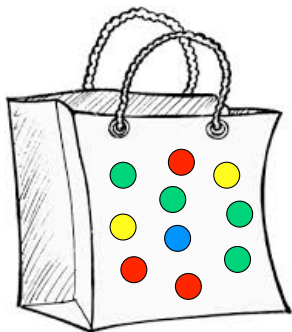
This leads to a definition!

Definition:

▶ An **event, E,** is a subset of outcomes: $E \subset \Omega$.

▶ The **probability of E** is defined as $Pr[E] = \sum_{\omega \in E} Pr[\omega]$.
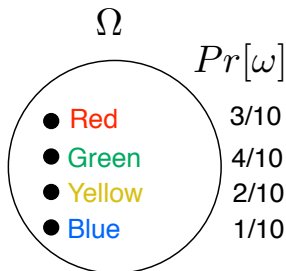

$$Pr[E] = \sum_{\omega \in E} Pr[\omega]$$

$$Pr[\omega] = \frac{1}{|\Omega|}$$

$$Pr[E] = \frac{|E|}{|\Omega|}$$

Event: Example

Physical experiment   Probability model

$\Omega = \{\text{Red, Green, Yellow, Blue}\}$
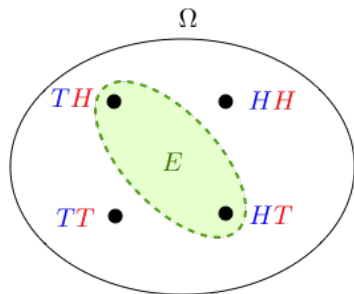$Pr[\text{Red}] = \dfrac{3}{10}, Pr[\text{Green}] = \dfrac{4}{10}$, etc.

$E = \{Red, Green\} \Rightarrow Pr[E] = \dfrac{3+4}{10} = \dfrac{3}{10} + \dfrac{4}{10} = Pr[\text{Red}] + Pr[\text{Green}].$

Probability of exactly one heads in two coin flips?
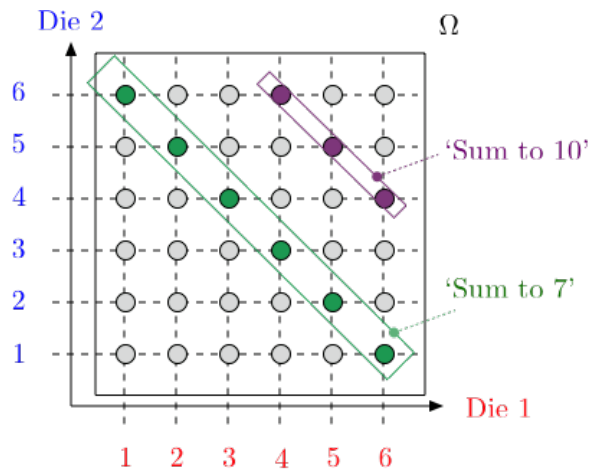
Sample Space, $\Omega = \{HH, HT, TH, TT\}$.

Uniform probability space: $Pr[HH] = Pr[HT] = Pr[TH] = Pr[TT] = \frac{1}{4}$.

Event, E, "exactly one heads": $\{TH, HT\}$.

$$Pr[E] = \sum_{\omega \in E} Pr[\omega] = \frac{|E|}{|\Omega|} = \frac{2}{4} = \frac{1}{2}.$$

Roll a red and a blue die.

$Pr[\text{Sum to 7}] = \dfrac{6}{36}$ $\qquad$ $Pr[\text{Sum to 10}] = \dfrac{3}{36}$

Example and Polls: 20 coin tosses.

20 coin tosses

Sample space: $\Omega =$ set of 20 fair coin tosses.
$\Omega = \{T, H\}^{20} \equiv \{0, 1\}^{20}; \quad |\Omega| = 2^{20}$.
What is more likely?

(A) $\omega_1 := (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$, or

(B) $\omega_2 := (1, 0, 1, 1, 0, 0, 0, 1, 0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 0, 0)$?

What is more likely?

(A) (E_1) Twenty Hs out of twenty, or

(B) (E_2) Ten Hs out of twenty?

Example: 20 coin tosses.

20 coin tosses

Sample space: $\Omega =$ set of 20 fair coin tosses.
$\Omega = \{T, H\}^{20} \equiv \{0, 1\}^{20}; \ |\Omega| = 2^{20}.$

- What is more likely?

    - $\omega_1 := (1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)$, or
    - $\omega_2 := (1,0,1,1,0,0,0,1,0,1,0,1,1,0,1,1,1,0,0,0)$?

    Answer: Both are equally likely: $Pr[\omega_1] = Pr[\omega_2] = \frac{1}{|\Omega|}$.

- What is more likely?

    (E_1) Twenty Hs out of twenty, or
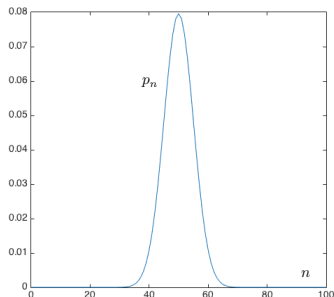 (E_2) Ten Hs out of twenty?

    Answer: Ten Hs out of twenty.

    Why? There are many sequences of 20 tosses with ten Hs;
    only one with twenty Hs. $\Rightarrow Pr[E_1] = \frac{1}{|\Omega|} \ll Pr[E_2] = \frac{|E_2|}{|\Omega|}$.

    $$|E_2| = \binom{20}{10} = 184,756.$$

Probability of n heads in 100 coin tosses.

$\Omega = \{H, T\}^{100}; \ |\Omega| = 2^{100}.$

Event E_n = 'n heads'; $|E_n| = \binom{100}{n}$

$p_n := Pr[E_n] = \frac{|E_n|}{|\Omega|} = \frac{\binom{100}{n}}{2^{100}}$

Observe:

▶ Concentration around mean:
  Law of Large Numbers;

▶ Bell-shape: Central Limit
  Theorem.

Exactly 50 heads in 100 coin tosses.

Sample space: $\Omega =$ set of 100 coin tosses $= \{H, T\}^{100}$.

$|\Omega| = 2 \times 2 \times \cdots \times 2 = 2^{100}$.

Uniform probability space: $Pr[\omega] = \frac{1}{2^{100}}$.

Event $E =$ "100 coin tosses with exactly 50 heads"

$|E|$?
Poll: (A) 100^50
(B) 2^50,
(C) $\binom{50}{100}$
(D) $\binom{100}{50}$
(E) IDK - TBH.

Choose 50 positions out of 100 to be heads.
$|E| = \binom{100}{50}$.

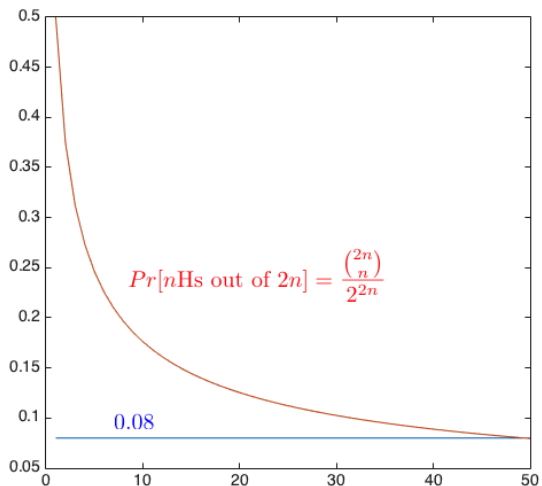$$Pr[E] = \frac{\binom{100}{50}}{2^{100}}.$$

Calculation.

(for large n):

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$

$$\binom{2n}{n} \approx \frac{\sqrt{4\pi n}(2n/e)^{2n}}{[\sqrt{2\pi n}(n/e)^n]^2} \approx \frac{4^n}{\sqrt{\pi n}}.$$

$$Pr[E] = \frac{|E|}{|\Omega|} = \frac{|E|}{2^{2n}} = \frac{1}{\sqrt{\pi n}} = \frac{1}{\sqrt{50\pi}} \approx .08.$$

Exactly 50 heads in 100 coin tosses.

$$Pr[n\text{Hs out of } 2n] = \frac{\binom{2n}{n}}{2^{2n}}$$

0.08

Summary.

1. Random Experiment
2. Probability Space: Ω; $Pr[\omega] \in [0,1]$; $\sum_\omega Pr[\omega] = 1$.
3. Uniform Probability Space: $Pr[\omega] = 1/|\Omega|$ for all $\omega \in \Omega$.
4. Event: "subset of outcomes." $A \subseteq \Omega$. $Pr[A] = \sum_{w \in A} Pr[\omega]$
5. Some calculations.