CS 70 Spring 2022 | Discrete Mathematics and Probability Theory
Satish Rao and Koushik Sen | HW 14

Due: Saturday 4/30, 4:00 PM
Grace period until Saturday 4/30, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Darts with Friends

Michelle and Alex are playing darts. Being the better player, Michelle's aim follows a uniform distribution over a disk of radius 1 around the center. Alex's aim follows a uniform distribution over a disk of radius 2 around the center.

(a) Let the distance of Michelle's throw from the center be denoted by the random variable X and let the distance of Alex's throw from the center be denoted by the random variable Y.

- What's the cumulative distribution function of X?
- What's the cumulative distribution function of Y?
- What's the probability density function of X?
- What's the probability density function of Y?

(b) What's the probability that Michelle's throw is closer to the center than Alex's throw? What's the probability that Alex's throw is closer to the center?

(c) What's the cumulative distribution function of $U = \max\{X,Y\}$?

(d) What's the cumulative distribution function of $V = \min\{X,Y\}$?

(e) What is the expectation of the absolute difference between Michelle's and Alex's distances from the center, that is, what is $\mathbb{E}[|X-Y|]$? [*Hint*: Use parts (c) and (d), together with the continuous version of the tail sum formula, which states that $\mathbb{E}[Z] = \int_0^\infty \mathbb{P}[Z \geq z]dz$.]

2 Continuous LLSE

Suppose that X and Y are uniformly distributed on the shaded region in the figure below.

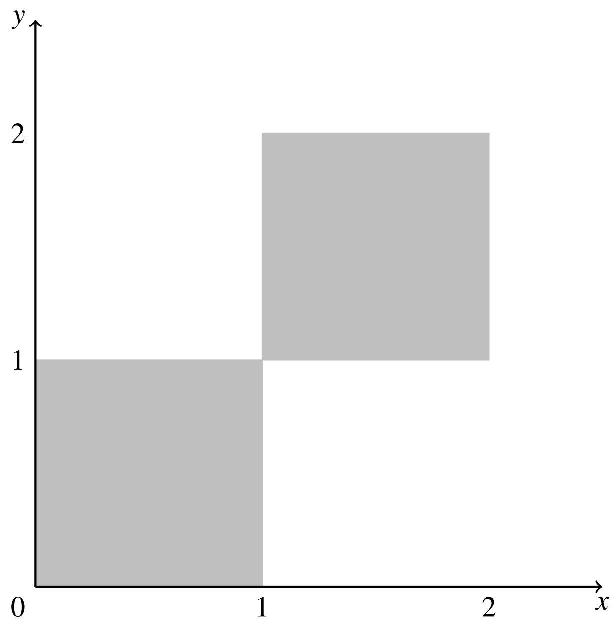

Figure 1: The joint density of (X,Y) is uniform over the shaded region.

That is, X and Y have the joint distribution:

$$f_{X,Y}(x,y) = \begin{cases} 1/2, & 0 \le x \le 1, 0 \le y \le 1 \\ 1/2, & 1 \le x \le 2, 1 \le y \le 2 \end{cases}$$

(a) Do you expect X and Y to be positively correlated, negatively correlated, or neither?

(b) Compute the marginal distribution of X.

(c) Compute $L[Y \mid X]$, the best linear estimator of Y given X.

(d) What is $\mathbb{E}[Y \mid X]$?

3 Chebyshev's Inequality vs. Central Limit Theorem

Let n be a positive integer. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with the following distribution:

$$\mathbb{P}[X_i = -1] = \frac{1}{12}; \qquad \mathbb{P}[X_i = 1] = \frac{9}{12}; \qquad \mathbb{P}[X_i = 2] = \frac{2}{12}.$$

(a) Calculate the expectations and variances of X_1, $\sum_{i=1}^n X_i$, $\sum_{i=1}^n (X_i - \mathbb{E}[X_i])$, and

$$Z_n = \frac{\sum_{i=1}^n (X_i - \mathbb{E}[X_i])}{\sqrt{n/2}}.$$

(b) Use Chebyshev's Inequality to find an upper bound b for $\mathbb{P}[|Z_n| \ge 2]$.

(c) Can you use b to bound $\mathbb{P}[Z_n \geq 2]$ and $\mathbb{P}[Z_n \leq -2]$?

(d) As $n \to \infty$, what is the distribution of Z_n?

(e) We know that if $Z \sim \mathcal{N}(0,1)$, then $\mathbb{P}[|Z| \leq 2] = \Phi(2) - \Phi(-2) \approx 0.9545$. As $n \to \infty$, can you provide approximations for $\mathbb{P}[Z_n \geq 2]$ and $\mathbb{P}[Z_n \leq -2]$?

4 Playing Blackjack

You are playing a game of Blackjack where you start with \$100. You are a particularly risk-loving player who does not believe in leaving the table until you either make \$400, or lose all your money. At each turn you either win \$100 with probability p, or you lose \$100 with probability $1-p$.

(a) Formulate this problem as a Markov chain; i.e. define your state space, transition probabilities, and determine your starting state.

(b) Compute the probability that you end the game with \$400.

5 Reflecting Random Walk

Alice starts at vertex 0 and wishes to get to vertex n. When she is at vertex 0 she has a probability of 1 of transitioning to vertex 1. For any other vertex i, there is a probability of $1/2$ of transitioning to $i+1$ and a probability of $1/2$ of transitioning to $i-1$.

(a) What is the expected number of steps Alice takes to reach vertex n? Write down the hitting-time equations, but do not solve them yet.

(b) Solve the hitting-time equations. [*Hint*: Let R_i denote the expected number of steps to reach vertex n starting from vertex i. As a suggestion, try writing R_0 in terms of R_1; then, use this to express R_1 in terms of R_2; and then use this to express R_2 in terms of R_3, and so on. See if you can notice a pattern.]

6 Boba in a Straw

Imagine that Jonathan is drinking milk tea and he has a very short straw: it has enough room to fit two boba (see figure).

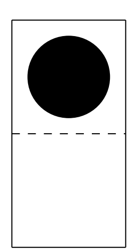

Figure 2: A straw with one boba currently inside. The straw only has enough room to fit two boba.

Here is a formal description of the drinking process: We model the straw as having two "components" (the top component and the bottom component). At any given time, a component can contain nothing, or one boba. As Jonathan drinks from the straw, the following happens every second:

1. The contents of the top component enter Jonathan's mouth.
2. The contents of the bottom component move to the top component.
3. With probability p, a new boba enters the bottom component; otherwise the bottom component is now empty.

Help Jonathan evaluate the consequences of his incessant drinking!

(a) Draw the Markov chain that models this process, and show that it is both irreducible and aperiodic.

(b) At the very start, the straw starts off completely empty. What is the expected number of seconds that elapse before the straw is completely filled with boba for the first time? [Write down the equations; you do not have to solve them.]

(c) Consider a slight variant of the previous part: now the straw is narrower at the bottom than at the top. This affects the drinking speed: if either (i) a new boba is about to enter the bottom component or (ii) a boba from the bottom component is about to move to the top component, then the action takes two seconds. If both (i) and (ii) are about to happen, then the action takes three seconds. Otherwise, the action takes one second. Under these conditions, answer the previous part again. [Write down the equations; you do not have to solve them.]

(d) Jonathan was annoyed by the straw so he bought a fresh new straw (same as the straw from Figure 1). What is the long-run average rate of Jonathan's calorie consumption? (Each boba is roughly 10 calories.)

(e) What is the long-run average number of boba which can be found inside the straw? [Maybe you should first think about the long-run distribution of the number of boba.]

(f) What is the long run probability that the amount of boba in the straw doesn't change from one second to the next?