CS 70   Discrete Mathematics and Probability Theory
Spring 2022   Koushik Sen and Satish Rao   (Optional) HW 6

Due: Saturday 3/5, 4:00 PM
Grace period until Saturday 3/5, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Counting, Counting, and More Counting

The only way to learn counting is to practice, practice, practice, so here is your chance to do so. Although there are many subparts, each subpart is fairly short, so this problem should not take any longer than a normal CS70 homework problem. You do not need to show work, and **Leave your answers as an expression** (rather than trying to evaluate it to get a specific number).

(a) How many ways are there to arrange n 1s and k 0s into a sequence?

(b) How many 7-digit ternary (0,1,2) bitstrings are there such that no two adjacent digits are equal?

(c) A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.

    i. How many different 13-card bridge hands are there?

    ii. How many different 13-card bridge hands are there that contain no aces?

    iii. How many different 13-card bridge hands are there that contain all four aces?

    iv. How many different 13-card bridge hands are there that contain exactly 6 spades?

(d) Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?

(e) How many 99-bit strings are there that contain more ones than zeros?

(f) An anagram of ALABAMA is any re-ordering of the letters of ALABAMA, i.e., any string made up of the letters A, L, A, B, A, M, and A, in any order. The anagram does not have to be an English word.

    i. How many different anagrams of ALABAMA are there?

ii. How many different anagrams of MONTANA are there?

(g) How many different anagrams of ABCDEF are there if:

i. C is the left neighbor of E

ii. C is on the left of E (and not necessarily E's neighbor)

(h) We have 9 balls, numbered 1 through 9, and 27 bins. How many different ways are there to distribute these 9 balls among the 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

(i) How many different ways are there to throw 9 identical balls into 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

(j) We throw 9 identical balls into 7 bins. How many different ways are there to distribute these 9 balls among the 7 bins such that no bin is empty? Assume the bins are distinguishable (e.g., numbered 1 through 7).

(k) There are exactly 20 students currently enrolled in a class. How many different ways are there to pair up the 20 students, so that each student is paired with one other student? Solve this in at least 2 different ways. **Your final answer must consist of two different expressions.**

(l) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a non-negative integer?

(m) How many solutions does $x_0 + x_1 = n$ have, if each x must be a *strictly positive* integer?

(n) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a *strictly positive* integer?

2 Grids and Trees!

Suppose we are given an $n \times n$ grid, for $n \geq 1$, where one starts at $(0,0)$ and goes to (n,n). On this grid, we are only allowed to move left, right, up, or down by increments of 1.

(a) How many shortest paths are there that go from $(0,0)$ to (n,n)?

(b) How many shortest paths are there that go from $(0,0)$ to $(n-1,n+1)$?

Now, consider shortest paths that meet the conditions where we can only visit points (x,y) where $y \leq x$. That is, the path cannot cross line $y = x$. We call these paths n-legal paths for a maze of side length n. Let F_n be the number of n-legal paths.

(c) Compute the number of shortest paths from $(0,0)$ to (n,n) that cross $y = x$. (Hint: Let (i,i) be the first time the shortest path crosses the line $y = x$. Then the remaining path starts from $(i,i+1)$ and continues to (n,n). If in the remainder of the path one exchanges y-direction moves with x-direction moves and vice versa, where does one end up?)

(d) Compute the number of shortest paths from $(0,0)$ to (n,n) that do not cross $y = x$. (You may find your answers from parts (a) and (c) useful.)

(e) A different idea is to derive a recursive formula for the number of paths. Fix some i with $0 \leq i \leq n-1$. We wish to count the number of n-legal paths where the last time the path touches the line $y = x$ is the point (i,i). Show that the number of such paths is $F_i \cdot F_{n-i-1}$. (Hint: If $i = 0$, what are your first and last moves, and where is the remainder of the path allowed to go?)

(f) Explain why $F_n = \sum_{i=0}^{n-1} F_i \cdot F_{n-i-1}$.

(g) Create and explain a recursive formula for the number of trees with n vertices $(n \geq 1)$, where each non-root node has degree at most 3, and the root node has degree at most 2. Two trees are different if and only if either left-subtree is different or right-subtree is different.

(Notice something about your formula and the grid problem. Neat!)

3 Fermat's Wristband

Let p be a prime number and let k be a positive integer. We have beads of k different colors, where any two beads of the same color are indistinguishable.

(a) We place p beads onto a string. How many different ways are there to construct such a sequence of p beads with up to k different colors?

(b) How many sequences of p beads on the string are there that use at least two colors?

(c) Now we tie the two ends of the string together, forming a wristband. Two wristbands are equivalent if the sequence of colors on one can be obtained by rotating the beads on the other. (For instance, if we have $k = 3$ colors, red (R), green (G), and blue (B), then the length $p = 5$ necklaces RGGBG, GGBGR, GBGRG, BGRGG, and GRGGB are all equivalent, because these are all rotated versions of each other.)

How many non-equivalent wristbands are there now? Again, the p beads must not all have the same color. (Your answer should be a simple function of k and p.)

[*Hint*: Think about the fact that rotating all the beads on the wristband to another position produces an identical wristband.]

(d) Use your answer to part (c) to prove Fermat's little theorem.

4 Counting on Graphs + Symmetry

(a) How many distinct undirected graphs are there with n labeled vertices? Assume that there can be at most one edge between any two vertices, and there are no edges from a vertex to itself. The graphs do not have to be connected.

(b) How many distinct cycles are there in a complete graph K_n with n vertices? Assume that cycles cannot have duplicated edges. Two cycles are considered the same if they are rotations or inversions of each other (e.g. (v_1, v_2, v_3, v_1), (v_2, v_3, v_1, v_2) and (v_1, v_3, v_2, v_1) all count as the same cycle).

(c) How many ways are there to color a bracelet with n beads using n colors, such that each bead has a different color? Note: two colorings are considered the same if one of them can be obtained by rotating the other.

(d) How many ways are there to color the faces of a cube using exactly 6 colors, such that each face has a different color? Note: two colorings are considered the same if one can be obtained from the other by rotating the cube in any way.

5 Proofs of the Combinatorial Variety

Prove each of the following identities using a combinatorial proof.

(a) For every positive integer $n > 1$,

$$\sum_{k=0}^{n} k \cdot \binom{n}{k} = n \cdot \sum_{k=0}^{n-1} \binom{n-1}{k}.$$

(b) For each positive integer m and each positive integer $n > m$,

$$\sum_{a+b+c=m} \binom{n}{a} \cdot \binom{n}{b} \cdot \binom{n}{c} = \binom{3n}{m}.$$

(Notation: the sum on the left is taken over all triples of nonnegative integers (a,b,c) such that $a+b+c=m$.)

6 Fibonacci Fashion

You have n accessories in your wardrobe, and you'd like to plan which ones to wear each day for the next t days. As a student of the Elegant Etiquette Charm School, you know it isn't fashionable to wear the same accessories multiple days in a row. (Note that the same goes for clothing items in general). Therefore, you'd like to plan which accessories to wear each day represented by subsets $S_1, S_2, \ldots, S_t$, where $S_1 \subseteq \{1,2,\ldots,n\}$ and for $2 \leq i \leq t$, $S_i \subseteq \{1,2,\ldots,n\}$ and S_i is disjoint from S_{i-1}.

(a) For $t \geq 1$, prove that there are F_{t+2} binary strings of length t with no consecutive zeros (assume the Fibonacci sequence starts with $F_0 = 0$ and $F_1 = 1$).

(b) Use a combinatorial proof to prove the following identity, which, for $t \geq 1$ and $n \geq 0$, gives the number of ways you can create subsets of your n accessories for the next t days such that no

accessory is worn two days in a row:

$$\sum_{x_1 \geq 0} \sum_{x_2 \geq 0} \cdots \sum_{x_t \geq 0} \binom{n}{x_1} \binom{n - x_1}{x_2} \binom{n - x_2}{x_3} \cdots \binom{n - x_{t-1}}{x_t} = (F_{t+2})^n.$$

(You may assume that $\binom{a}{b} = 0$ whenever $a < b$.)