CS 70 Spring 2022 — Discrete Mathematics and Probability Theory — Koushik Sen and Satish Rao — HW 03

Due: Saturday, 2/12, 4:00 PM
Grace period until Saturday, 2/12, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Planarity and Graph Complements

Let $G = (V,E)$ be an undirected graph. We define the complement of G as $\overline{G} = (V,\overline{E})$ where $\overline{E} = \{(i,j)|i,j \in V, i \neq j\} - E$; that is, $\overline{G}$ has the same set of vertices as G, but an edge e exists is $\overline{G}$ if and only if it does not exist in G.

(a) Suppose G has v vertices and e edges. How many edges does $\overline{G}$ have?

(b) Prove that for any graph with at least 13 vertices, G being planar implies that $\overline{G}$ is non-planar.

(c) Now consider the converse of the previous part, i.e., for any graph G with at least 13 vertices, if $\overline{G}$ is non-planar, then G is planar. Construct a counterexample to show that the converse does not hold.

Hint: Recall that if a graph contains a copy of K_5, then it is non-planar. Can this fact be used to construct a counterexample?

2 Touring Hypercube

In the lecture, you have seen that if G is a hypercube of dimension n, then

- The vertices of G are the binary strings of length n.
- u and v are connected by an edge if they differ in exactly one bit location.

A *Hamiltonian tour* of a graph is a sequence of vertices $v_0, v_1, \ldots, v_k$ such that:

- Each vertex appears exactly once in the sequence.

- Each pair of consecutive vertices is connected by an edge.
- v_0 and v_k are connected by an edge.

(a) Show that a hypercube has an Eulerian tour if and only if n is even. (*Hint: Euler's theorem*)

(b) Show that every hypercube has a Hamiltonian tour.

3 Connectivity

Consider the following claims regarding connectivity:

(a) Prove: If G is a graph with n vertices such that for any two non-adjacent vertices u and v, it holds that $\deg u + \deg v \geq n-1$, then G is connected.

[*Hint:* Show something more specific: for any two non-adjacent vertices u and v, there must be a vertex w such that u and v are both adjacent to w.]

(b) Give an example to show that if the condition $\deg u + \deg v \geq n-1$ is replaced with $\deg u + \deg v \geq n-2$, then G is not necessarily connected.

(c) Prove: For a graph G with n vertices, if the degree of each vertex is at least $n/2$, then G is connected.

(d) Prove: If there are exactly two vertices with odd degrees in a graph, then they must be in the same connected component (meaning, there is a path connecting these two vertices).

[*Hint:* Proof by contradiction.]

4 Graph Coloring

Prove that a graph with maximum degree at most k is $(k+1)$-colorable.

5 Modular Practice

Solve the following modular arithmetic equations for x and y.

(a) $9x+5 \equiv 7 \pmod{11}$.

(b) Show that $3x+15 \equiv 4 \pmod{21}$ does not have a solution.

(c) The system of simultaneous equations $3x+2y \equiv 0 \pmod 7$ and $2x+y \equiv 4 \pmod 7$.

(d) $13^{2019} \equiv x \pmod{12}$.

(e) $7^{21} \equiv x \pmod{11}$.

6 Nontrivial Modular Solutions

(a) What are all the possible perfect cubes modulo 7?

(b) Show that any solution to $a^3 + 2b^3 \equiv 0 \pmod 7$ must satisfy $a \equiv b \equiv 0 \pmod 7$.

(c) Using part (b), prove that $a^3 + 2b^3 = 7a^2b$ has no non-trivial solutions (a,b) in the integers. In other words, there are no integers a and b, that satisfy this equation, except the trivial solution $a = b = 0$.

[*Hint:* Consider some nontrivial solution (a,b) with the smallest value for $|a|$ (why are we allowed to consider this?). Then arrive at a contradiction by finding another solution (a',b') with $|a'| < |a|$.]

7 Check Digits: ISBN

In this problem, we'll look at a real-world applications of check-digits.

International Standard Book Numbers (ISBNs) are 10-digit codes $(d_1 d_2 \dots d_{10})$ which are assigned by the publisher. These 10 digits contain information about the language, the publisher, and the number assigned to the book by the publisher. Additionally, the last digit d_{10} is a "check digit" selected so that $\sum_{i=1}^{10} i \cdot d_i \equiv 0 \pmod{11}$. (*Note that the letter X is used to represent the number 10 in the check digit.*)

(a) Suppose you have a very worn copy of the (recommended) textbook for this class. You want to list it for sale online but you can only read the first nine digits: 0-07-288008-? (the dashes are only there for readability). What is the last digit? Show your work.

(b) Wikipedia says that you can determine the check digit by computing $\sum_{i=1}^{9} i \cdot d_i \pmod{11}$. Show that Wikipedia's description is equivalent to the above description.

(c) Prove that changing any single digit of the ISBN will render the ISBN invalid. That is, the check digit allows you to *detect* a single-digit substitution error.

(d) Can we ever switch two distinct digits in an ISBN number and get another valid ISBN number? For example, could 01<u>2</u>34<u>5</u>678X and 01<u>5</u>34<u>2</u>678X both be valid ISBNs? Explain.

8 Wilson's Theorem

Wilson's Theorem states the following is true if and only if p is prime:

$$(p-1)! \equiv -1 \pmod p.$$

Prove both directions (it holds if AND only if p is prime).

Hint for the if direction: Consider rearranging the terms in $(p-1)! = 1 \cdot 2 \dots \cdot p-1$ to pair up terms with their inverses, when possible. What terms are left unpaired?

Hint for the only if direction: If p is composite, then it has some prime factor q. What can we say about $(p-1)! \pmod q$?